Detection Theory

R.J. Marks II Class Notes

Professor Gary Wise

Texas Tech University (1974)

$x = e^{\frac{x}{\pi}} = 1.73989895$

$x = e^{1/x} = 1.763222834$

$x = e^{\tanh x} = 2.693638367$

$x = \tanh e^{x} = 0.99089283\overline{25}$

A brief note on probability spaces

A random experiment having a finite number of outcomes offers little conceptual difficulty. In this case we can assign probabilities to each of the outcomes and to any combination of the outcomes. Combinations of the experimental outcomes form events. For example, the experiment may be the rolling of a die, and the outcomes may correspond to the six faces of the die. The events may be any combination of the six faces, and with each of these events, we can associate a probability. However, in the general case, it is neither possible nor desirable to assign probabilities to all combinations of outcomes. We will want to consider unions, intersections, sequences, and limits of events, and we want to be sure that the resulting sets are also events, i.e., that probabilities are assigned to them. We will now outline a brief axiomatic foundation of probability theory.

A set is a collection of arbitrary elements. All sets will be sets of elements of a fixed nonempty set Ω. The "empty set" will be denoted by $\emptyset$. A set, whose elements are sets, will be called a class of sets. When set operations performed on the sets in a class $\mathcal{A}$ give as a result sets which also belong to $\mathcal{A}$, then the class $\mathcal{A}$ is said to be closed under those set operations.

A field $\mathcal{F}$ is a nonempty class of sets which is closed under complementation and finite unions. Thus a field $\mathcal{F}$ is a nonempty class of sets of Ω such that

1.) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

2.) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

These two properties are sufficient to define a field. A number of other properties follow immediately. For example,

3.) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $AB \in \mathcal{F}$.

4.) $\Omega \in \mathcal{F}$.

5.) $\emptyset \in \mathcal{F}$.

6.) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A - B \in \mathcal{F}$.

All of the set operations can be performed any finite number of times on the members of the field $\mathcal{F}$ without obtaining a set not in $\mathcal{F}$.

A field $\mathcal{B}$ on Ω is called a σ-field (sigma field) or Borel field on Ω if it is closed under countable unions. Thus a Borel field $\mathcal{B}$ is a field on Ω such that if $A_1, A_2, A_3, \ldots \in \mathcal{B}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$.

It follows that a Borel field is a nonempty class closed under all countable set operations. The elements of the Borel field will be the events of the random experiment.

A probability space is defined as the triple $(\Omega, \mathcal{B}, P)$, where Ω is the sample space, $\mathcal{B}$ is a Borel field on Ω, and P is a real-valued set function such that:

1.) For every event $A \in \mathcal{B}$, $P(A) \geq 0$.

2.) $P(\Omega) = 1$.

3.) If $A_1, A_2, A_3, \ldots$ is any countable sequence of mutually disjoint events in $\mathcal{B}$, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots\ldots) = \sum_{i=1}^{\infty} P(A_i).$$

Notice that probabilities are only defined for events belonging to the Borel field. In applications, all "sensible" collections of outcomes will be events.

Example: The sample space Ω may correspond to the six faces of a die. The Borel field $\mathcal{B}$ may be taken as all possible combinations of the six faces, i.e., there will be $2^6 = 64$ sets in the Borel field. The probability measure P may assign the value $1/6$ to each of the six faces, i.e. each of the elementary events.

The elements of the probability space are not uniquely specified by the underlying experiment. For example, consider the rolling of a die. Mr. X may model the experiment as in the above example. However, Mr. Y may wish to bet only on "odd" or "even." He may then argue that there are only two outcomes, forming the space $\Omega = \{\text{even, odd}\}$, and the events are four: Ω, $\emptyset$, $\{\text{even}\}$, $\{\text{odd}\}$. According to Mr. X, the event $\{\text{even}\}$ consists of the three elements corresponding to the three even faces of the die. But according to Mr. Y, the event $\{\text{even}\}$ consists of the single element "even".

see: Thomas — <u>An Introduction to Applied Probability and Random Processes</u>, Chap. 2

Papoulis — <u>Probability, Random Variables, and Stochastic Processes</u>, Chap. 2.

HW #1

EE 6321

#1. A random variable is distributed according to a Cauchy distribution

$$f(x) = \frac{m}{\pi\,(m^2 + x^2)}$$

The parameter m can take on either of two equally likely values m_0 and m_1, $m_0 < m_1$. Design a statistical test to decide on the basis of a single measurement the correct value of m. The test should be such that the probability of error is minimized. The test should be reduced to a simplified form.

#2. We wish to test for the presence of a signal on the basis of one measurement.

$$H_0 : \; y = n \qquad\qquad f(n) = \frac{1}{\sqrt{2\pi}}\, e^{-n^2/2}$$

$$H_1 : \; y = 2 + n$$

We assign a cost of one unit to an error of the first kind and a cost of 5 units to an error of the second kind. There is no cost for a correct decision. We wish to design a statistical test using the minimax criterion. Design the detector. (It should be reduced to a simplified form.)

EE 6321 HW #1

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EE 6321
H.W. #1
DUE: 9-15-75

100

BOB MARKS

1. $f(x) = \dfrac{m}{\pi(m^2+x^2)}$ $m>0$

(FOR $f(x)$ TO BE A PDF, $m>1$)

H_0: $m = m_0$; $p_0(x) = \dfrac{m_0}{\pi(m_0^2+x^2)}$

H_1: $m = m_1$; $p_1(x) = \dfrac{m_1}{\pi(m_1^2+x^2)}$

GIVEN: $P[H_0] = P[H_1] = 0.5 \Rightarrow \pi_1 = \pi_0 = 0.5$

$m_0 < m_1$

$$\Lambda(x) = \frac{p_1(x)}{p_0(x)} = \frac{m_1(m_0^2+x^2)}{m_0(m_1^2+x^2)}$$

SO TEST IS

$$\Lambda(x) = \frac{m_1(m_0^2+x^2)}{m_0(m_1^2+x^2)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \qquad \left(= \frac{\pi_0}{\pi_1}\right)$$

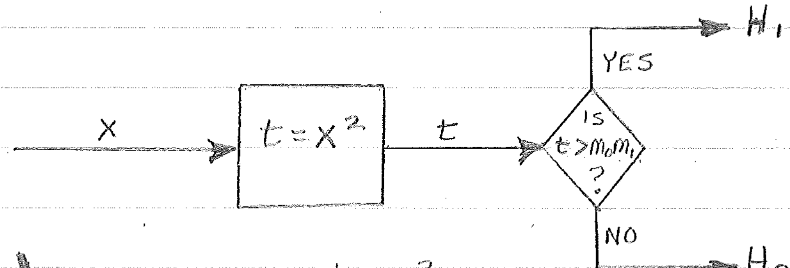
$$m_0^2 + x^2 \gtrless \frac{m_0}{m_1}(m_1^2+x^2)$$

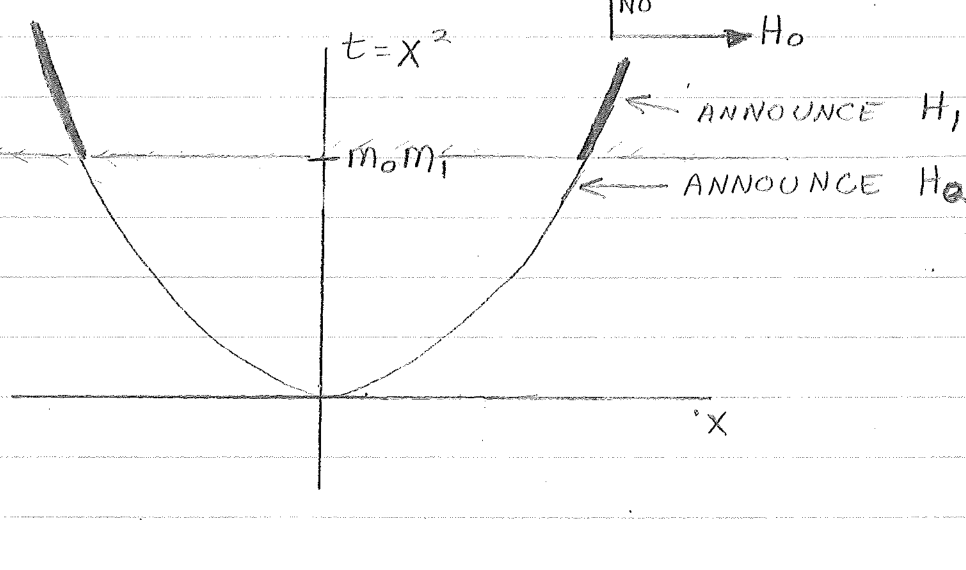
$$x^2\left[1 - \frac{m_0}{m_1}\right] \gtrless m_0 m_1 - m_0^2$$

$$x^2\left[\frac{m_1-m_0}{m_1}\right] \gtrless m_0(m_1-m_0) \qquad ; \quad m_1 - m_0 > 0$$

$$x^2 \gtrless m_0 m_1$$

OR $$x^2 \underset{H_0}{\overset{H_1}{\gtrless}} m_0 m_1$$





EE 6321
H.W.#1
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$\boxed{100}$

BOB MARKS

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(FOR $f(x)$ TO BE A PDF, $m > 1$)

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H_1: $m = m_1$; $p_1(x) = \dfrac{m_1}{\pi(m_1^2 + x^2)}$

GIVEN: $P[H_0] = P[H_1] = 0.5 \Rightarrow \pi_1 = \pi_0 = 0.5$

$m_0 < m_1$

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$$\Lambda(x) = \frac{m_1(m_0^2 + x^2)}{m_0(m_1^2 + x^2)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \qquad \left(= \frac{\pi_0}{\pi_1}\right)$$

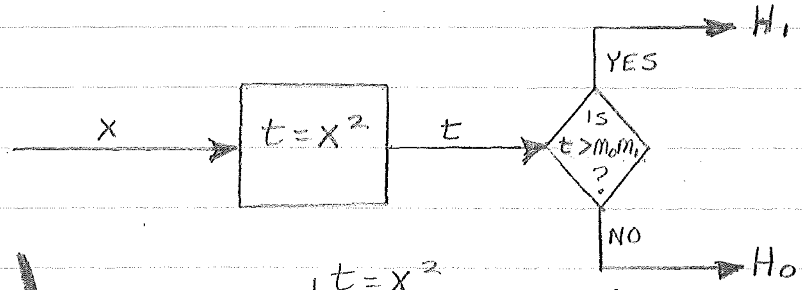
$$m_0^2 + x^2 \gtrless \frac{m_0}{m_1}(m_1^2 + x^2)$$

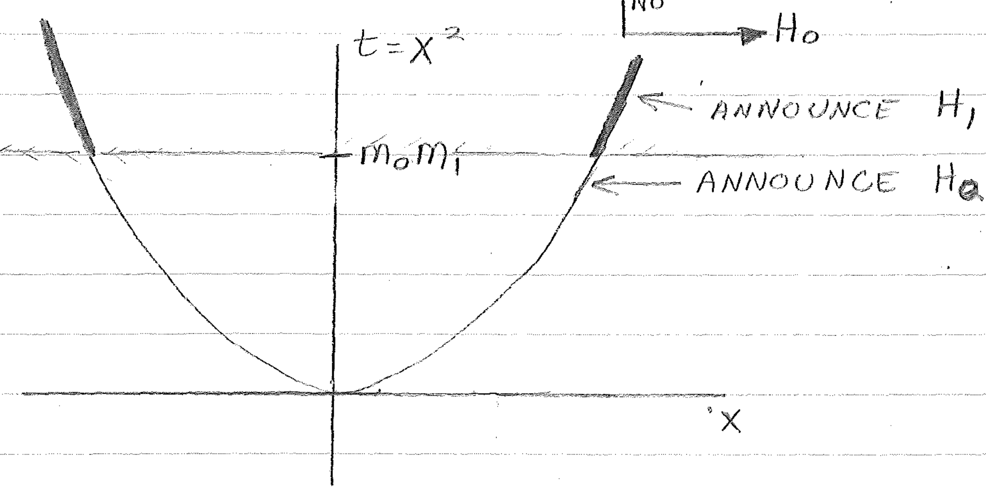
$$x^2\left[1 - \frac{m_0}{m_1}\right] \gtrless m_0 m_1 - m_0^2$$

$$x^2\left[\frac{m_1 - m_0}{m_1}\right] \gtrless m_0(m_1 - m_0) \qquad ; \; m_1 - m_0 > 0$$

$$x^2 \gtrless m_0 m_1$$

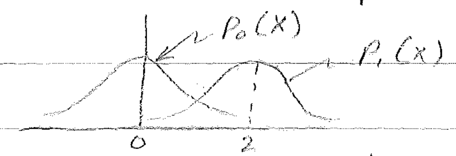
OR $\quad x^2 \underset{H_0}{\overset{H_1}{\gtrless}} m_0 m_1$





2. $H_0 : Y = n \qquad C_0 = 1 \qquad P_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$H_1 : Y = n+2 \qquad C_1 = 5 \qquad P_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}}$



BAYES RISK FOR $C_{00} = C_{11} = 0$

$\mathcal{R}(\pi_1) = [C_1 Q_1 - C_0 Q_0]\pi_1 + C_0 Q_0$

FOR $C_{11} = C_{00} = 0$, $\mathcal{R}(0) = \mathcal{R}(1) = 0$

$\Rightarrow \mathcal{R}_{MAX} = \mathcal{R}(\pi_{MAX}) \ni 0 < \pi_{MAX} < 1$

$\mathcal{R}(\hat{\pi}, \pi_1) = [C_1 \hat{Q}_1 - C_0 \hat{Q}_0]\pi_1 + C_0 \hat{Q}_0 = \text{CONST.}$

$\Rightarrow$ MINIMAX RISK $= C_0 \hat{Q}_0 \qquad (= C_1 \hat{Q}_1)$

AND $C_1 \hat{Q}_1 = C_0 \hat{Q}_0$

$\hat{Q}_1 = \int_{R_0} P_1(Y)\,dY = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} dx = F(x_0 - 2)$

$\hat{Q}_0 = \int_{R_1} P_0(Y)\,dY = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 - F(x_0)$

$\Rightarrow C_1 \hat{Q}_1 = 5F(x_0 - 2)$

$= C_0 \hat{Q}_0 = 1 - F(x_0)$

INTUITIVELY, $x_0 < 1$

$x_0 = 0.47 \Rightarrow 1 - F(x_0) = 1 - F(0.47) = 0.3192$
AND $5F(x_0 - 2) = 5F(-1.53) = 5[1 - F(1.53)] = 0.3150$
A DIFFERENCE OF 0.0042

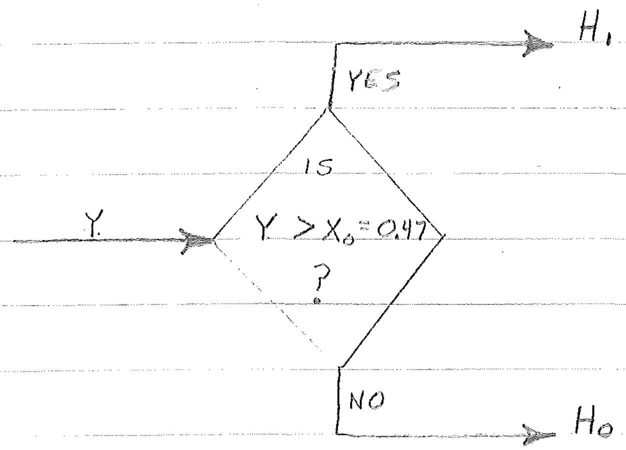
$x_0 = 0.48 \Rightarrow 1 - F(x_0) = [1 - F(0.48)] = 0.3156$
AND $5F(x_0 - 2) = 5F(-1.52) = 5[1 - F(1.52)] = 0.3215$
A DIFFERENCE OF 0.0059

$x_0 = 0.46 \Rightarrow 1 - F(x_0) = 1 - F(0.46) = 0.3228$
AND $5F(x_0 - 2) = 5F(-1.54) = 5[1 - F(1.54)] = 0.3080$
A DIFFERENCE OF 0.0148

$\therefore$ TO TWO PLACES, $X_0 = 0.47$

AND THE MINIMAX RISK (TO TWO PLACES) IS 0.32

THE DETECTOR IS



EE 6321 HW #2

One measure of performance is the receiver operating characteristic (ROC). The ROC is a graph of detection probability (β) versus false alarm probability (α). The quantity β is plotted on the y axis and α on the x axis.

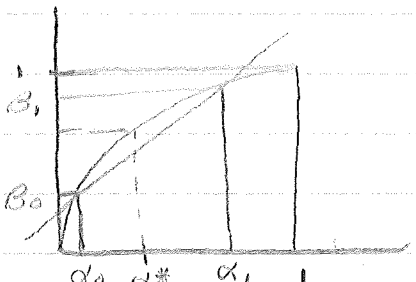
#1. Show that the ROC for a Neyman - Pearson test cannot go beneath the straight line connecting the point $(0,0)$ with the point $(1,1)$.

#2. Show that the ROC for a Neyman-Pearson test must be concave.

BOB MARKS
H.W. # 2
DUE 9/26/75

$\boxed{80}$

1,2. PROVE THE ROC OF THE LRT (VIA N.P) IS CONCAVE. DENOTE THE ROC CURVE BY $B(\alpha)$.



but you didn't take it into account!

ASSUMING RANDOMIZATION (IF ANY) IS TAKEN INTO ACCOUNT,
LET C_i FOR THE POINTS α_i, B_i BE DEFINED FROM THE N.P. TEST $\phi_i(x)$, $i=0,1$

$$\phi_i(x) = \begin{cases} 1, & \Lambda(x) > C_i \\ 0, & \Lambda(x) < C_i \end{cases}$$

see back

WLOG, LET $\alpha_1 > \alpha_0 \Rightarrow C_1 \leq C_0$. (randomization can increase α)

$B(\alpha)$ IS CONCAVE, IFF FOR EVERY SECANT LINE S_{01} THROUGH THE POINTS (α_0, B_0), (α_1, B_1),

$S_{01} \leq B(\alpha) \quad \forall \quad \alpha \in [\alpha_0, \alpha_1]$.

ASSUME, CONTRARILY, $\exists \ \alpha^* \in [\alpha_0, \alpha_1] \ni S_{01} > B(\alpha$

This won't work, in general.

DEFINE THE TEST $\phi^*(x)$ BY

$$\phi^*(x) = \begin{cases} 0, & \Lambda < C_1 \\ \frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0}, & C_1 \leq \Lambda \leq C_0 \\ 1, & \Lambda > C_0 \end{cases}$$

what if $C_1 =$

THEN:

$$B = E_1[\phi^*(x)] = \left[\frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0}\right](B_1 - B_0) + B_0$$
$$\alpha = E_0[\phi^*(x)] = \left[\frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0}\right](\alpha_1 - \alpha_0) + \alpha_0 = \alpha$$

You ... considering the boundary probabilities ...

NOTE THAT $\phi^*(x)$ IS DEFINED ONLY OVER THE INTERVAL $\alpha_0 \leq \alpha \leq \alpha_1$, SINCE OUTSIDE THESE LIMITS, $\frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0} < 0$ OR > 1.

THE ROC FOR $\phi^*(x)$ IS THUS GIVEN BY

$$\beta - \beta_0 = \frac{\beta_1 - \beta_0}{\alpha_1 - \alpha_0}(\alpha - \alpha_0) \quad ; \alpha_0 \leq \alpha \leq \alpha_1$$

THIS IS THE RELATIONSHIP FOR THE SECANT LINE L_{01}. THUS, IF $\exists\ \alpha^* \in [\alpha_0, \alpha_1]$ $\ni S_{01} > \beta(\alpha)$, THEN $\phi^*(x)$ IS MORE POWERFUL THAN $\phi(x)$ (THE N-P LRT). THIS IS CONTRARY TO THE N-P LEMMA. THUS, $\beta(\alpha)$ IS CONCAVE. Q.E.D.

PROBLEM 1: SINCE $\beta(\alpha)$ IS CONCAVE AND PASSES THROUGH THE POINTS $(0,0)$ AND $(1,1)$, $\beta(\alpha) \geq \alpha\ \forall\ \alpha$. i.e., ALL VALUES OF $\beta(\alpha)$ LIE ABOVE OR ON THE LINE PASSING THROUGH THE PTS. $(0,0)$, $(1,1)$

DUE MON

EE 6321 HW #3

Consider the following detection problem:

$H_0 : \quad y = n$

$H_1 : \quad y = 1 + n$

We have one observation.

Assume that the noise has the density $\frac{1}{\sqrt{2\pi}} e^{-n^2/2}$.
Design the Neyman-Pearson detection scheme for $\alpha = 0.01$.
Call this detector (*). Calculate β for this detector.

Now consider the case where the noise has the density

$\frac{1}{2} e^{-|n|}$.

Using detector (*), i.e. detector designed for Gaussian noise, calculate α and β for this different noise.

BOB MARKS
H.W. # 3
6 OCT. 75.

$\boxed{100}$

1. $H_0: Y = n$; $P_{0*}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$H_1: Y = n+1$; $P_{1*}(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}$

$p(n) = \frac{1}{\sqrt{2\pi}} e^{-n^2/2}$

GIVEN $\alpha^* = 0.01 = \int_T^\infty P_{0*}(x)dx = 1 - F(T)$

WHERE $F(T) = \int_{-\infty}^T P_0(x)dx$

$1 - F(T) = 0.01 \Rightarrow T^* = 2.33 \Rightarrow t \underset{H_0}{\overset{H_1}{\gtrless}} 2.33$ ✓

$\beta^* = \int_T^\infty P_{*1}(x)dx = \int_{2.33}^\infty \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx$

$\xi = x - 1 \Rightarrow x = 2.33 \Rightarrow \xi = 1.33$

$\beta^* = \int_{1.33}^\infty \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi = 1 - F(1.33) = 0.0918$ ✓

FOR LAPLACE NOISE USING DETECTOR (*)

$P_0(x) = \frac{1}{2} e^{-|x|}$

$P_1(x) = \frac{1}{2} e^{-|x-1|}$

$\alpha = \int_T^\infty P_0(x)dx = \int_{2.33}^\infty \frac{1}{2} e^{-x} dx$

$= +\frac{1}{2} e^{-2.33} = 0.0486$ ✓ ← BIGGER

$\beta = \int_T^\infty P_1(x)dx = \int_{2.33}^\infty \frac{1}{2} e^{-(x-1)} dx$

$= \int_{1.33}^\infty \frac{1}{2} e^{-x} dx$

$= \frac{1}{2} e^{-1.33} = .132$ ✓ ← ALSO BIGGER

EE 6321

HW #4

$$H_0: \; y(t) = n(t)$$
$$H_1: \; y(t) = s(t) + n(t) \qquad a \le t \le b$$

Noise is zero mean, Gaussian, with continuous, positive definite autocorrelation function $R(t_1, t_2)$.
λ_k and $\phi_k(t)$ are the eigenvalues and orthonormal eigenfunctions of R.
$s(t) \in L_2$. $\qquad s_k = (s, \phi_k)$

Define $d_K^2 = \sum_{k=1}^{K} \frac{s_k^2}{\lambda_k}$.

#1. Show that, if $\lim_{K \to \infty} d_K^2 = \infty$, then for any positive $\alpha < 1$ and any number $b < 1$, there is always a test such that the false alarm probability is α and the detection probability is greater than b. (SINGULAR DETECTION) SEE HELSTROM

#2. Assume $R(t_1, t_2) = e^{-|t_1 - t_2|}$. Consider the detector which calculates $G = \int_a^b e^{-t} y(t)\, dt$ and then decides $G \underset{H_0}{\overset{H_1}{\gtrless}} T$. If this is the Neyman-Pearson detector, then, subject to a scale factor, what is the signal $s(t)$?

ROB MARKS
DUE 11/3/75

100

H.W. #4

$H_0 : Y(t) = n(t)$

$H_1 : Y(t) = s(t) + n(t)$

$$d_K^2 = \sum_{K=1}^{K} S_K^2/\lambda_K$$

1. IF $\lim_{K\to\infty} d_K^2 \to \infty$, THEN SHOW $\forall$ $\alpha<1$ & $b<1$, $\exists$ A TEST $\phi(x)$ $\ni$ FOR A GIVEN α, $\beta > b$.

DETECTOR FOR (ZERO-MEAN GAUSSIAN w/ POS. DEF. $R(t,s)$) WAS

$$\sum_{K=1}^{K} \frac{S_K Y_K}{\lambda_K} \gtrless_{H_0}^{H_1} \ln \Lambda_0 + \sum_{K=1}^{K} \frac{S_K^2}{2\lambda_K}$$

OR EQUIVALENTLY

$$G_K \gtrless_{H_0}^{H_1} \ln \Lambda_0 + \frac{1}{2} d_K^2$$

WHERE $G_K = \sum_{K=1}^{K} \frac{S_K Y_K}{\lambda_K}$

CORRESPONDING TEST, $\phi_K(x)$, IS

$$\phi_K(x) = \begin{cases} 1 & ; G_K > \ln\Lambda_0 + \frac{1}{2} d_K^2 \\ P & ; G_K = \ln\Lambda_0 + \frac{1}{2} d_K^2 \\ 0 & ; G_K < \ln\Lambda_0 + \frac{1}{2} d_K \end{cases}$$

SINCE $G_K \sim N$ (ie IS NORMALLY DISTRIBUTI

$P[G_K = \text{CONSTANT}] = 0$.

THUS, WLOG:

$$\phi_K(x) = \begin{cases} 1 \; ; \; G_K > \ln \Lambda_0 + \frac{1}{2} d_K^2 \\ 0 \; ; \; G_K \leq \ln \Lambda_0 + \frac{1}{2} d_K^2 \end{cases}$$

NOW $\quad E_0[G_K] = E_0\left[\sum_{k=1}^{K} \frac{s_k y_k}{\lambda_k}\right] = 0$

$$E_1[G_K] = \sum_{k=1}^{K} s_k^2/\lambda_k = d_K^2$$

$$VAR_0[G_K] = VAR_1[G_K]$$

$$= \sum_{k=1}^{K} s_k^2/\lambda_k = d_K^2$$

THUS $\quad G_K \sim N(d_K^2, d_K^2)$ under H_1.

$$\alpha = E_0[\phi(x)] = \frac{1}{\sqrt{2\pi}\, d_K} \int_{\ln \Lambda_0 + \frac{1}{2} d_K^2}^{\infty} e^{-\frac{y^2}{2 d_K^2}} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln \Lambda_0}{d_K} + \frac{1}{2} d_K}^{\infty} e^{-y^2/2} \, dy$$

$$= 1 - \Phi\left[\frac{\ln \Lambda_0}{d_K} + \frac{1}{2} d_K\right]$$

$$\beta = E_1[\phi(x)] = \frac{1}{\sqrt{2\pi}\, d_K} \int_{\ln \Lambda_0 + \frac{1}{2} d_K^2}^{\infty} e^{-\frac{(y - d_K^2)^2}{2 d_K^2}} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln \Lambda_0}{d_K} - \frac{1}{2} d_K}^{\infty} e^{-y^2/2} \, dy$$

$$= 1 - \Phi\left[\frac{\ln \Lambda_0}{d_K} - \frac{1}{2} d_K\right]$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

FIXING α FIXES $\ln \Lambda_0$:

$$\ln \Lambda_0 = d_K \Phi^{-1}(1-\alpha) - \tfrac{1}{2} d_K^2$$

THUS

$$\beta = 1 - \Phi\left[\Phi^{-1}(1-\alpha) - d_K\right] \quad \checkmark$$

NOW: $d_K^2 = \sum_{n=1}^{K} \frac{S_K^2}{\lambda_K}$

$\lambda_K = E\left[X_K^2\right] > 0$

$\Rightarrow d_K^2 > 0 \quad ; \quad d_K > 0 \Rightarrow \lim_{K \to \infty} d_K = \infty$ ✓

(d_K = STD. DEV. OF $G_K > 0$)

d_K IS MONOTONIC WITH K AND HAS NO UPPER BOUND (ie $\lim_{K \to \infty} d_K = \infty$).

$\Phi^{-1}(X)$ IS MONOTONICALLY INCREASING ON THE INTERVAL $(0,1)$ AND HAS NO LOWER BOUND. THUS,

$\forall\ \alpha \in (0,1)\ \exists\ K$ AND A b_K RELATED BY

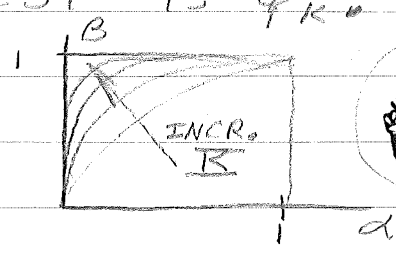
$$b_K = 1 - \Phi\left[\Phi^{-1}(1-\alpha) - d_K\right]$$

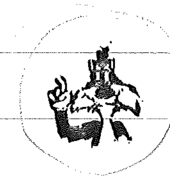
WHERE b_K IS MONOTONICALLY INCREASING WITH K. $[b_K \in (0,1)]$

THUS, $\forall\ \beta \in (0,1)\ \exists\ K \ni b_K > \beta$.

THE CORRESPONDING TEST IS ϕ_K.

NOTE: THE R.O.C. CURVE GETS "MORE CONCAVE" WITH INCREASING K ✓





2. GIVEN

$$R(t_1, t_2) = e^{-|t_1 - t_2|}$$

$$G = \int_a^b e^{-t} Y(t)\,dt$$

N.P. DETECTOR: $G \gtrless_{H_0}^{H_1} T$

SINCE

$$G = \int_a^b Y(t) g(t)\,dt$$

APPARENTLY, THE PSEUDO SIGNAL MAY BE TAKEN AS

$$g(t) = e^{-t}\ ;\ a<t<b \quad (g(t) \text{ SPECIFIES } s(t))$$

THE SIGNAL $s(t)$ AND PSEUDO SIGNAL, $g(t)$, ARE RELATED BY

$$s(t) = \int_a^b R(t, \tau) g(\tau)\,d\tau$$

THUS:

$$s(t) = \int_a^b e^{-|t-\tau|} e^{-\tau}\,d\tau$$

DEFINE:

$$\mathrm{rect}(\xi) = \begin{cases} 1 & ;\ |x| \le \frac{1}{2} \\ 0 & ;\ \text{OTHERWISE} \end{cases}$$

$2W = b - a \Leftarrow$ INTERVAL WIDTH

$p = \frac{b+a}{2} \Leftarrow$ INTERVAL MIDPOINT

THEN

$$s(t) = \int_{-\infty}^{\infty} e^{-|t-\tau|} e^{-\tau}\,\mathrm{rect}\left[\frac{\tau - p}{2W}\right] d\tau$$

$$= e^{-|t|} * \left[e^{-t}\,\mathrm{rect}\left(\frac{t-p}{2W}\right)\right] d\tau$$

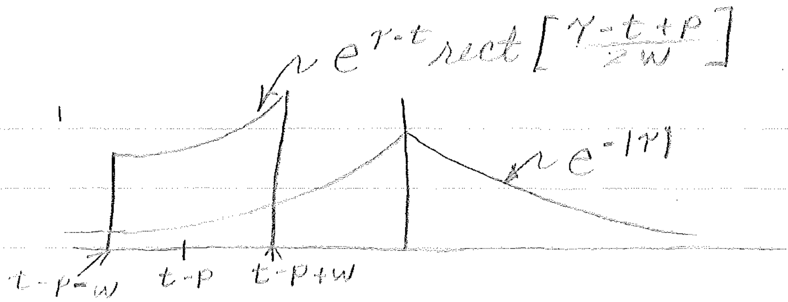
WHERE "$*$" DENOTES CONVOLUTION.

SINCE CONVOLUTION IS COMMUTATIVE:

$$s(t) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-(t-\tau)}\,\mathrm{rect}\left[\frac{t-\tau-p}{2W}\right] d\tau$$

$$\mathrm{rect}(x) = \mathrm{rect}(-x)$$

$$\Rightarrow s(t) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{\tau - t}\,\mathrm{rect}\left[\frac{\tau - t + p}{2W}\right] d\tau$$



(1) LIMITS: $t-P+W<0 \Rightarrow t<P-W \Rightarrow t<\frac{b+a}{2}-\frac{b-a}{2}$

$\Rightarrow t<a$

$$\begin{aligned}
\therefore s(t) &= \int_{t-P-W}^{t-P+W} e^{\tau-t}e^{\tau}\,d\tau \\
&= e^{-t}\int_{t-P-W}^{t-P+W} e^{2\tau}\,d\tau \\
&= e^{-t}\ \tfrac{1}{2}e^{2\tau}\Big|_{t-P-W}^{t-P+W} \\
&= \tfrac{1}{2}e^{-t}\left[e^{2(t-P+W)}-e^{2(t-P-W)}\right] \\
&= \tfrac{1}{2}e^{-t}e^{2(t-P)}\left[e^{2W}-e^{-2W}\right] \\
&= e^{-t+2t-2P}\sinh 2W \\
&= e^{+t}\ e^{-(b+a)}\sinh(b-a)
\end{aligned}$$

(2) LIMITS: $t-P-W<0<t-P+W \Rightarrow -P-W<-t<-P+W$

$\Rightarrow P+W>t>P-W \Rightarrow P-W<t<P+W$

$\Rightarrow \frac{b+a}{2}-\frac{b-a}{2}<t<\frac{b+a}{2}+\frac{b-a}{2}$

$\Rightarrow a<t<b$

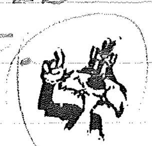
$$\begin{aligned}
\therefore s(t) &= \int_{t-P-W}^{0} e^{2\tau-t}\,d\tau + \int_{0}^{t-P+W} e^{-t}\,d\tau \\
&= e^{-t}\left[\tfrac{1}{2}e^{2\tau}\Big|_{t-P-W}^{0} + (t-P+W)\right] \\
&= e^{-t}\left[\tfrac{1}{2}\left(1-e^{2(t-P-\omega)}\right)+(t-P+\omega)\right] \\
&= \tfrac{1}{2}e^{-t}-\tfrac{1}{2}e^{t-2(P-\omega)}+(t-P+\omega)e^{-t} \\
&= \left(\tfrac{1}{2}+t+\omega-P\right)e^{-t}-\tfrac{1}{2}e^{-2(P+\omega)}e^{+t} \\
&= \left(\tfrac{1}{2}+t-a\right)e^{-t}-\tfrac{1}{2}e^{-2b}e^{t}
\end{aligned}$$

It would have been ~~easi~~ easier to just directly evaluate the integral. ETC.

THE N-P. DETECTION SCHEME ONLY LOOKS AT $Y(t)$ (AND $S(t)$) OVER THE INTERVAL $a < t < b$. THUS ALL WE NEED TO KNOW IS

$$s(t) = \left(\tfrac{1}{2} + t - a\right)e^{-t} - \tfrac{1}{2}e^{-2b}e^{t} \quad ; \; a < t < b$$

THIS RELATIONSHIP MAY DIFFER FROM THE TRUE SIGNAL ($S_t(t)$) BY A CONSTANT, WHICH, IN THE DETECTOR DESIGN, WOULD BE ABSORBED BY THE THRESHOLD T.



BOB MARKS

EXAM #1

EE 6321

#1.

$$H_0 : y = n$$
$$H_1 : y = s + n$$

The noise has a Laplace density, i.e. $f(n) = \frac{\gamma}{2} e^{-\gamma |n|}$.
The signal s is a positive constant. We have one observation.

Consider the test : $y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s}{2}$.

A. For this test, what is α ?

B. For this value of α, what is the Neyman-Pearson test ?

C. Assuming the hypotheses are equally likely, $\pi_0 = \pi_1 = \frac{1}{2}$ what is the test that minimizes the probability of error ?

#2. Consider the locally optimal Neyman-Pearson detector for a positive signal in stationary white Laplace noise. Assume we take three observations. What is the smallest positive value of α such that the resulting test is not randomized?

#3. Consider testing for a positive constant signal in stationary white Gaussian noise.

$H_0 : \; y_i = n_i$

$H_1 : \; y_i = s + n_i$

$$f(n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-n^2/2\sigma^2}$$

Show that for a fixed positive value of α, β can be made arbitrarily close to one by taking a sufficiently large number of observations.

92

BOB MARKS
TEST # 1
10-13-75

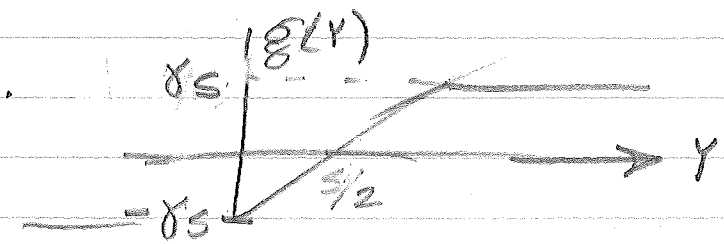
1. a. $\alpha = \int_{S/2}^{\infty} \frac{\gamma}{2} e^{-\gamma|Y|} dY$ (TEST IS HARD LIMITER)

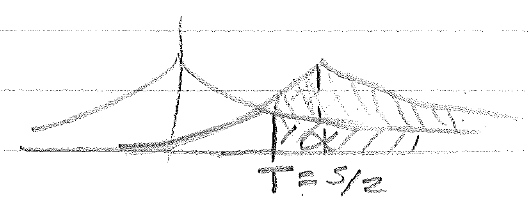
$$= \begin{cases} \cancel{\frac{\gamma}{2}\frac{1}{\gamma} e^{\gamma Y}\Big|_{S/2}^{\infty}} & ; \ Y < 0 \\ -\frac{\gamma}{2}\frac{1}{\gamma} e^{-\gamma Y}\Big|_{S/2}^{\infty} & ; \ Y > 0 \end{cases}$$

BUT $S/2 > 0$

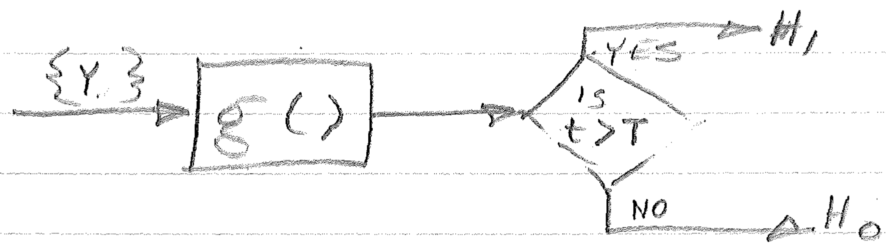
$$\Rightarrow \alpha = -\frac{1}{2} e^{-\gamma Y}\Big|_{S/2}^{\infty}$$
$$= -\frac{1}{2}\left[e^{-\infty} - e^{-\gamma S/2}\right]$$
$$= \frac{1}{2} e^{-\gamma S/2}$$

b.





NEYMANN PEARSON DETECTOR:



−8

$\Rightarrow$ N.P. TEST IS

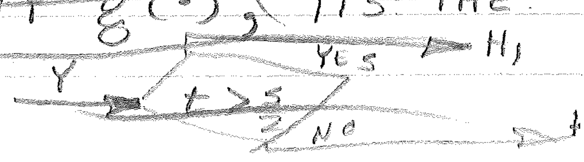
$$g(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \cancel{\ln\frac{S}{2}} \; T$$

What is α for this detector? (hint: its NOT $\frac{1}{2}e^{\ldots}$)

w/ nonlinearity, same as part A. →

~~WITHOUT NONLINEARITY~~ $g(\cdot)$, ~~ITS THE SAME AS PART a:~~

for this particular value of α.



c. $\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\pi_0}{\pi_1}$

$\Lambda(Y) = e^{-\gamma[|Y-s|-|Y|]} \gtrless_{H_0}^{H_1} \frac{\pi_0}{\pi_1}$

$[|Y-s|+|Y|] \gtrless_{H_0}^{H_1} \frac{1}{\gamma} \ln \frac{\pi_0}{\pi_1}$

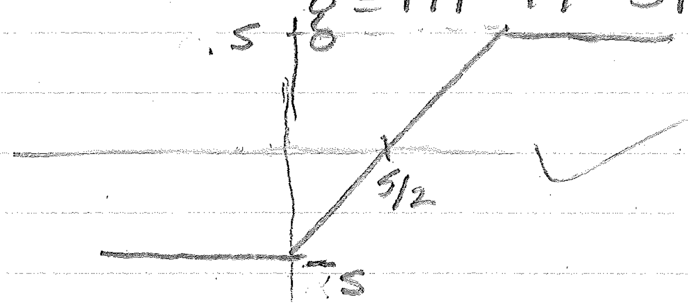
⇒ USING g AS IN (b) GIVES TEST

$\Rightarrow g(Y) \gtrless_{H_0}^{H_1} \ln \frac{\pi_0}{\pi_1}$

BUT $\pi_0 = \pi_1 = \frac{1}{2} \Rightarrow \ln \frac{\pi_0}{\pi_1} = 0$

$-|Y-s| + |Y| \gtrless_{H_0}^{H_1} 0$

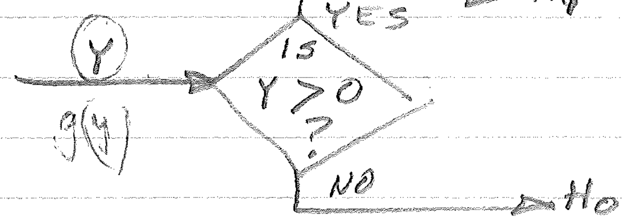
$g = |Y| - |Y-s|$



MERELY DETECT THE SIGN OF Y
g(y)

OK

THE ZMN MAY THUS BE REDUCED BACK TO A HARD LIMITER



2. $f(x) = \frac{\gamma}{2} e^{-\gamma |x|}$

$$\sum_{i=1}^{3} -\frac{d}{dY} \ln \frac{\gamma}{2} e^{-\gamma |Y_i|} \gtrless T_1$$

$$\sum +\frac{d}{dY} \gamma |Y| \gtrless T_1$$

$$t = \sum_{i=1}^{3} \operatorname{sgn} Y_i \gtrless T$$

$P_0[t = 3] = \frac{1}{8}$

$P_0[t = 1] = \frac{3}{8}$

$P_0[t = -1] = 3/8$

$P_0[t = -3] = 1/8$

$$\phi = \begin{cases} 1 & t > T \\ \rho & t = T \\ 0 & t < T \end{cases}$$

FOR $\alpha = 1/8$ LET $T \in (1, 3)$

$\Rightarrow \alpha = \frac{1}{8} \cdot 1 + 0 \cdot \frac{5}{8} + 0 \cdot 0 = \frac{1}{8}$

$\therefore$ SMALLEST $\alpha \neq 0$ NOT REQUIRING RANDOMIZATION IS $\alpha = 1/8$

3. $P_0 = \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{x^2}{2K\sigma^2}}$

GIVEN α, T IS GIVEN BY

$$T = \sqrt{K}\,\sigma\, \Phi^{-1}(1-\alpha) \qquad (Eq. 1)$$

$$\ni \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$$

$$P_1 = \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{(x-KS)^2}{2K\sigma^2}}$$

$$\beta = \int_T^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\left(\frac{x-KS}{2K\sigma^2}\right)^2} dx$$

$$= \int_{\frac{T-KS}{\sqrt{K}\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 1 - \Phi\left[\frac{T-KS}{\sqrt{K}\,\sigma}\right]$$

better way is plug in Eq(1) here, use monotonicity of $\Phi(\cdot)$, etc.

$$\Rightarrow \frac{T-KS}{\sqrt{K}\,\sigma} = \Phi^{-1}[1-\beta]$$

$$T = \sqrt{K}\,\sigma\,\Phi^{-1}(1-\beta) + KS$$

EQUATING WITH EQ 1

$$\cancel{K}\sigma^2[\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)]^2 = K^{\cancel{2}}S^2$$

$$K = \frac{\sigma^2}{S^2}[\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)]^2 \qquad Eq\ 2$$

HOLD σ, S, AND α CONSTANT.

$\Phi^{-1}(0) = -\infty$, $\Phi^{-1}(1) = +\infty$

$\Phi^{-1}(\gamma)$ IS MONOTONICALLY INCREASING WITH γ

$\Rightarrow [\text{Const.} - \Phi^{-1}(1-\beta)]^2$ IS " INCREASING WITH β

$\therefore$ DUE TO BOUNDS ON $\Phi^{-1}(1-\beta)$, $\forall\, \beta \in (0,1)$

$\exists\, K \geq 0$ TO SOLVE EQN. 2

AND $\lim_{K\to\infty} \beta = 1$.

OK – kind of sloppy

EE 6321 EXAM #2

#1. We wish to reject or accept a quantity of devices depending on whether the average lifetime is less than or greater than some standard. We base our decision on k iid observations of device lifetime. The lifetime is modeled as a random variable with density

$$p(x,\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0, \quad \theta > 0,$$

where θ is the average lifetime. If $\theta \leq t$, we reject the devices. If $\theta > t$, we accept them. Design a test such that the probability of accepting bad devices is some given number α, and the probability of accepting good devices is high. (You need not evaluate the threshold. When you specify the test, be sure to specify when to reject and when to accept.)

#2. We wish to test for the presence of a decaying exponential signal in zero mean stationary Gaussian noise. The waveform is sampled slowly enough that the noise samples are independent. However, there is a problem in synchronizing the clock at the receiver. The situation is modeled as follows:

$$H_0 : \quad y_k = n_k$$

$$k = 1, 2, \cdots, K.$$

$$H_1 : \quad y_k = e^{-t_k} + n_k$$

The noise is iid Gaussian $\sim N(0, \sigma^2)$. Due to a synchronization problem, the sampling instants $\{t_k\}$ are modeled as $t_k = kT - \theta$, where θ is an unknown parameter that lies in the interval $\left[-\frac{T}{10}, \frac{T}{10}\right]$. It is desired to design a detector that has a given false alarm probability and a high detection probability. Design it. (You need not evaluate the threshold.)

#3. Consider the continuous time detection of a sure signal $s(t)$ in zero mean Gaussian noise with a continuous, positive definite autocorrelation function $R(t_1, t_2)$. Assume the finite observation interval is $[a, b]$. The fidelity criterion is detector performance, i.e. for a fixed α, get a large β.

Discuss the problem of choosing a "good" signal $s(t)$, subject to an energy constraint, i.e.

$$\int_a^b [s(t)]^2 \, dt \leq \mathcal{E} .$$

Does there exist a "best" signal? If so, what is it; if not, why not?

85

1. $p(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad x \geq 0 \; , \; \theta > 0$

$$p(x;\theta) = \underset{\substack{\| \\ 1}}{h(x)} \; \underset{\substack{\| \\ \frac{1}{\theta}}}{C(\theta)} \; \exp\left[\underset{\substack{\| \\ x}}{T(x)} \; \underset{\substack{\| \\ -1/\theta}}{Q(\theta)}\right]$$

$\Rightarrow$ *Monotone likelihood Ratio*

$H_0 : \theta \leq t$

$H_1 : \theta > t$

May thus use test

$$\phi(x) = \begin{cases} 1 & ; \; x > x_0 \\ p & ; \; x = x_0 \\ 0 & ; \; x < x_0 \end{cases}$$

Wrong way

$$\ln p(x;\theta) = \ln\frac{1}{\theta} - \frac{x}{\theta} = -\ln\theta - \frac{x}{\theta}$$

$$\frac{d}{d\theta}\ln p(x;\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2} = \frac{x-\theta}{\theta^2}$$

?

$\Rightarrow X$ IS MAXIMUM LIKELIKHOOD

$E[X] = \theta \Leftarrow$ UNBIASED (FOR $k=1$)

$$p(x_1, x_2, x_3 \ldots ; \theta) = \left(\frac{1}{\theta}\right)^k e^{-\frac{1}{\theta}\sum_{k=1}^{K} x_k}$$

IN SIMILAR ARGUMENT (*Monotone likelihood Ratio*)

$\sum_{k=1}^{K} X_K$ *is sufficient statistic*

also know $\frac{1}{K} \sum_{K=1}^{K} X_K = \overline{X}$ (sample mean) has property in exponential distribution that
$$E[\overline{X}] = \Theta \Leftarrow \text{unbiased}$$
so let $\bar{x}$ be suffic. statistic

$$\frac{d}{d\Theta} \ln p(x_1, \ldots x_K; \Theta) = \frac{d}{d\Theta} \ln \frac{1}{\Theta^K}$$
$$- \frac{1}{\Theta} \sum_{K=1}^{K} X_K$$
$$\frac{d}{d\Theta}\Big[K \ln \Theta\Big] - \frac{K}{\Theta} \overline{X} \Big]$$
$$= \frac{-K}{\Theta} + \frac{K\overline{X}}{\Theta^2}$$
$$= \frac{-K\Theta + K\overline{X}}{\Theta^2} = \frac{K[\overline{X} - \Theta]}{\Theta^2}$$

$\overline{X}$ is efficient & achieves C-R lower bound. Test is then (Using $\phi(x) = \begin{cases} 1 ; & \overline{X} > T \\ 0 ; & \overline{X} < T \end{cases}$

OK

$$\overline{X} = \frac{1}{K} \sum_{K=1}^{K} X_K \underset{H_0}{\overset{H_1}{\gtrless}} T \checkmark$$

OR

$$\sum_{K=1}^{K} X_K \underset{H_0}{\overset{H_1}{\gtrless}} \tilde{T} \checkmark$$

but I don't like your reasoning

Actually, I think you got the answer through a large amount of luck. But, you got the answer.

$$\alpha = \sup_{\Theta < t} E_0[\phi(x)]$$

Test is UMP (no involvement with Θ -
[$\bar{x}$ computation involves K fold convolution of $p(x;\Theta)$]

2. $H_0 : Y_K = n_K$

$H_1 : Y_K = e^{-t_K} + n_K$

$$= e^{-kT+\theta} + n_K$$

$$\theta \in \left[-\frac{T}{10}, \frac{T}{10}\right]$$

$$p_0(\vec{Y}_K) = \prod_{K=1}^{K} \frac{1}{\sqrt{2\pi\lambda_K}} e^{-\frac{Y_K^2}{2\lambda_K}}$$

$$p_1(\vec{Y}_K) = \prod_{K=1}^{K} \frac{1}{\sqrt{2\pi\lambda_K}} e^{-\frac{(Y_K - e^{-t_K})^2}{2\lambda_K}}$$

$$\ln \Lambda(\vec{Y}_K) = \sum_{K=1}^{K} -\frac{(Y_K - e^{-t_K})^2}{2\lambda_K} + \frac{Y_K^2}{2\lambda_K}$$

$$2\lambda_K \ln \Lambda(\vec{Y}_K) = \sum_{K=1}^{K} -(Y_K - e^{-t_K})^2 + Y_K^2$$

$$= \sum +2Y_K e^{-t_K} - e^{-2t_K} \cancel{+Y_K^2}$$

$$= \sum_{K=1}^{K} e^{-t_K}\left[2Y_K - e^{-t_K}\right]$$

$$\cancel{e^{-t_K} > 0}$$

$$\cancel{e^{t_K}\, 2\lambda_K \ln \Lambda(\vec{Y}_K) = 2Y_K}$$

$$= \sum_{K=1}^{K} e^{-(kT-\theta)}\left[2Y_K - e^{-(kT-\theta)}\right]$$

$$(\qquad) e^{-\theta} = \sum_{K=1}^{K} e^{-kT}\left[2Y_K - e^{-kT} e^{\theta}\right]$$

$$= \sum_{K=1}^{K} 2Y_K e^{-kT} - \sum_{K=1}^{K} e^{-2kT} e^{\theta}$$

$$2\lambda_K \ln \Lambda(\vec{Y}_K) = \sum_{K=1}^{K} 2Y_K e^{-t_K} - e^{-2t_K}$$

$$= \sum_{K=1}^{K} e^{-t_K}\left[2Y_K - e^{-t_K}\right]$$

$$= \sum_{K=1}^{K} e^{-KT+\theta}\left[2Y_K - e^{-KT+\theta}\right]$$

$$e^{-\theta}(\qquad) = \sum_{K=1}^{K} e^{-KT}\left[2Y_K - e^{-KT+\theta}\right]$$

$$= \sum_{K=1}^{K} 2Y_K e^{-KT} - \sum_{K=1}^{K} e^{-2KT} e^{\theta}$$

~~Don't use ...~~ ~~$\hat{\theta}_{ML} = \ldots$~~

~~$P_1(x, \theta)$~~

$$\tfrac{1}{2} e^{-\theta} 2\lambda_K \ln \Lambda(Y_K) + \tfrac{1}{2}\sum_{K=1}^{K} e^{-2KT} e^{\theta} = \sum_{K=1}^{K} Y_K e^{-KT}$$

looks like a UMP ~~test~~ :

$$\sum_{K=1}^{K} Y_K e^{-KT} \gtrless_{H_0}^{H_1} \tilde{T} = \text{THRESHOLD}$$

$$\alpha = E_0[\phi(x)]$$

3. $R(t_1, t_2)$ $\quad s_2(t) = \int R(t_1, t_2) q(t_2) dt_2$

$[a, b]$

$$SNR = \frac{\int_a^b s(t)q(t)dt}{E\left[\int_a^b n(t)q(t)dt\right]}$$

The best signal would be when

$$\int_a^b [s(t)]^2 dt = \mathcal{E} = \sum_{K=1}^{\infty} S_K^2$$

This will maximize the signal to noise ratio:

$$d^2 = \sum_{K=1}^{\infty} \frac{S_K^2}{\lambda_K} = \sum_{K=1}^{\infty} q_K S_K = \int_a^b q(t)s(t)dt$$

Cont. time detector is

$$\int_a^b Y(t)q(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0 + \frac{1}{2}d^2$$

The ~~test~~ statistic ~~has~~

$$G = \int_a^b Y(t)q(t)dt$$

has

$$E_0[G] = 0 \qquad E_1[G] = d^2 \quad \mathrm{var}_{0,1}[G] = d^2$$

Maximizing d^2 for a given α, will give ~~mean~~ maximum β. (ie β is monotonic w/R/T d^2)

this, effectively, is the problem!

-15

Close

EE 6321 FINAL EXAM

#1. A simple two-hypothesis problem is:

$$H_0: \quad f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$H_1: \quad f_1(x) = \frac{1}{2} e^{-|x|} .$$

There is one observation.

A. Find the N-P optimal test.

B. Show clearly the possible decision regions.

#2. Let the density under H_0 be $N(0, \sigma_0^2)$ where σ_0^2 is given. Under H_1 let the density be $N(0, \sigma_1^2)$, where $\sigma_1^2 > \sigma_0^2$ but is otherwise unknown. There are K iid observations. Decide whether or not a UMP test exists. If it does, find it. If it does not, show explicitly why not.

#3. A sequence of K independent observations $X_1, X_2, \ldots, X_K$
15 ~~are~~ available. The testing hypotheses are

$$H_0 : \; X_i = n_i$$
$$H_1 : \; X_i = S + n_i$$

where S is a positive constant. If the n_i have the common density function

$$f(x) = K_1 \exp\left[-K_2 |x|^3\right], \quad -\infty < x < \infty .$$

find the locally optimal detector for the signal s.

$$-\frac{\delta}{\delta x} \ln f(x)$$

#4. The problem is
$$H_0 : \; x = n$$
$$H_1 : \; x = 1 + n$$

where $f(n) = e^{-2|n|}$. There is one observation.

A. Consider the test $x \underset{H_0}{\overset{H_1}{\gtrless}} 3/4$. What are α and β

B. Consider the N-P test for the above problem. What is the smallest positive value of α for which the test is not randomized?

#5. We wish to test for the presence of a time varying signal in white stationary Gaussian noise.

$$H_0 : \quad x_k = n_k$$

$$k = 1, 2, \cdots, K$$

$$H_1 : \quad x_k = \theta s_k + n_k$$

$$n_k \quad \text{iid} \quad \sim N(0, \sigma^2) .$$

Design a detector which for a given α will yield a high value of β. Consider two cases:

θ is unknown, and θ is unknown but positive.

Marks

$\boxed{98}$ very good!

1. a. $\Lambda = \frac{f_1}{f_0}$

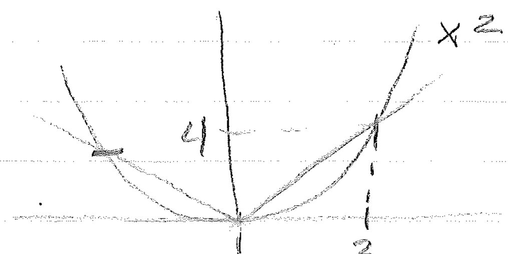
$$= \frac{\frac{1}{2} e^{-|x|}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

$$= \frac{\sqrt{2\pi}}{2} e^{-|x| + \frac{x^2}{2}}$$

$$\frac{2}{\sqrt{2\pi}} \Lambda = e^{-|x| + \frac{x^2}{2}}$$

$$\ln(\quad) = -|x| + \frac{x^2}{2}$$

$$2\ln(\quad) = x^2 - 2|x| \underset{H_0}{\overset{H_1}{\gtrless}} T$$



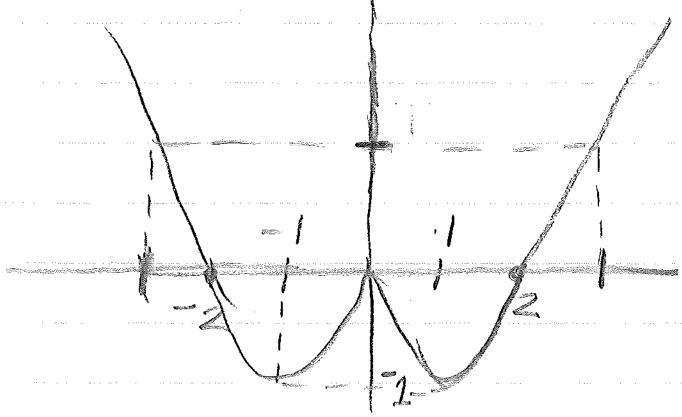
$x^2 = 2x$

$x = 2$

$x^2 - 2x$

$2x - 2 = 0$

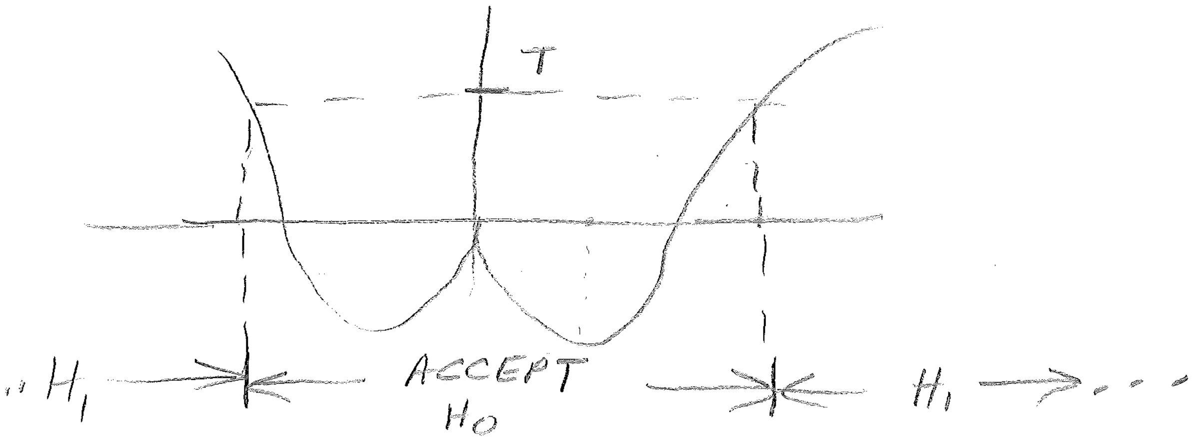
MIN @ $\Rightarrow x = 1$



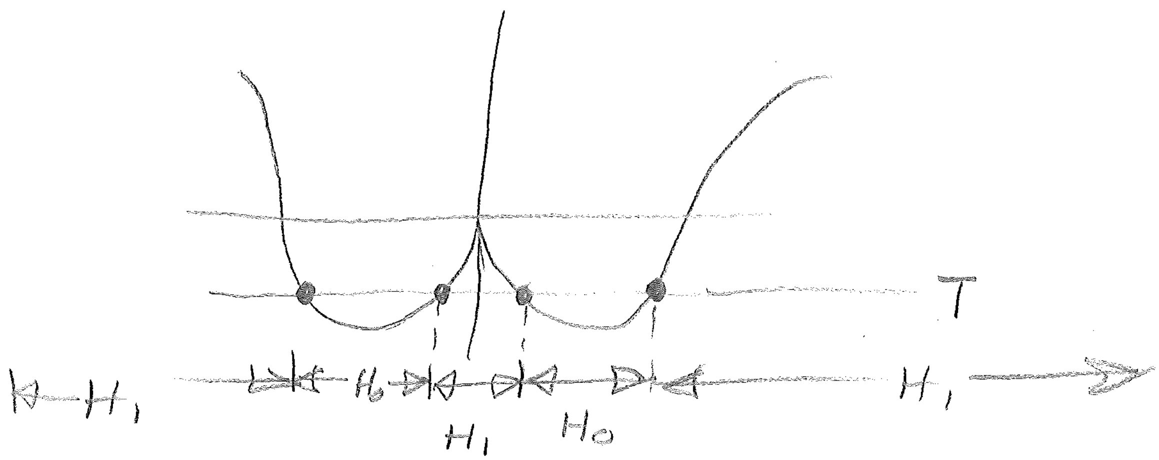
CONT. $\longrightarrow$

b. Basically (3) cases

CASE (1) $T > 0$



(2) $-1 < T < 0$



(3) FOR $T < -1$,
<u>always</u> announce H_1

Marbo

2. $H_0 : \sigma_0^2$ KNOWN

$H_1 : \sigma_1^2 > \sigma_0^2$

-2

K iid

$$P_0(x) = \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{x_k^2}{2\sigma_0^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^K e^{-\frac{\sum_{k=1}^{K} x_k^2}{2\sigma_0^2}}$$

$$P_1(x) = \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-x_k^2/2\sigma_1^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma_1}\right)^K e^{-\sum_{k=1}^{K} x_k^2/2\sigma_1^2}$$

$$\Lambda = \frac{P_1}{P_0} = \left(\frac{\sigma_0}{\sigma_1}\right)^K e^{-\sum_{k=1}^{K} x_k^2 \left[\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right]} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$

$$? \sum x_k^2 \left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right] \underset{H_0}{\overset{H_1}{\gtrless}} (\quad)$$

GIVEN $\sigma_1^2 > \sigma_0^2$

$$\Rightarrow \frac{1}{\sigma_1^2} < \frac{1}{\sigma_0^2} \Rightarrow \frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2} < 0$$

$$\Rightarrow \sum x_k^2 \underset{H_1}{\overset{H_0}{\gtrless}} T$$

~~NOTE REVERSAL~~

IS A UMP TEST

(i.e., WE DON'T NEED TO KNOW σ_1^2 TO DESIGN TEST)

3

Marks

3. $H_0 : x_i = n_i$

$H_1 : x_i = s + n_i$

$$f(x_i) = K_1 e^{-K_2 |x|^3}$$

$$f(\vec{X}_K) = \prod_{i=1}^{K} K_1 e^{-K_2 |X_K|^3}$$

$$= (K_1)^K e^{-K_2 \sum_{i=1}^{K} |X_K|^3}$$

The locally optimum ZNL for independent observations:

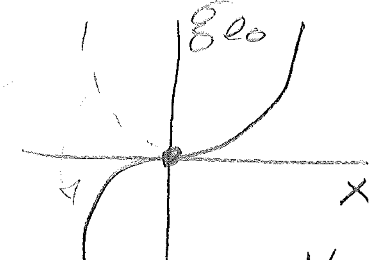
$$g_{lo}(x) = -\frac{\delta}{\delta x} \ln f(x)$$

$$g_{lo}(x) = -\frac{\delta}{\delta x_i} \ln f(x)$$

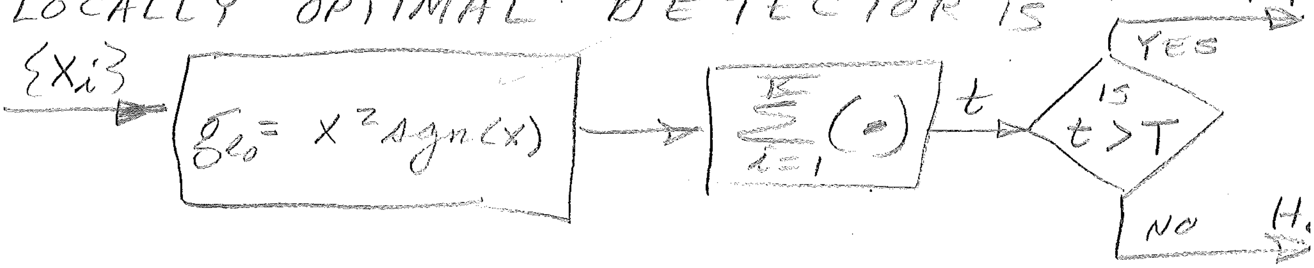
$$= K_2 \frac{d}{dx} |x|^3$$

$$= K_2\, x^2 \, \mathrm{sgn}(x) \Rightarrow$$

$$\sim x^2 \, \mathrm{sgn}(x)$$



LOCALLY OPTIMAL DETECTOR IS



(BY LOCALLY OPTIMAL, WE MEAN WE ARE MAXIMIZING $\frac{\delta \beta}{\delta s}\Big|_{s=0^+}$)

Marks

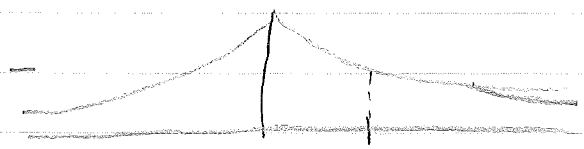
4. $H_0: x = n$
$H_1: x = n+1$

$$f(n) = e^{-2|n|}$$

a. $x \underset{H_0}{\overset{H_1}{\gtrless}} 3/4$

$$\phi(x) = \begin{cases} 1 & ; x > 3/4 \\ 0 & x < 3/4 \end{cases}$$

(NO RANDOMIZATION necessary here)



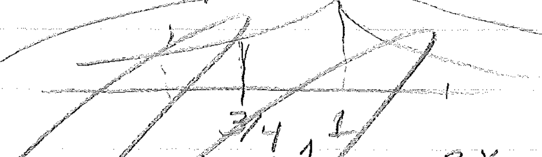
3/4

$$P_0[x > \tfrac{3}{4}] = \int_{3/4}^{\infty} e^{-2x}\,dx$$

$$= -\tfrac{1}{2} e^{-2x} \Big|_{3/4}^{\infty}$$

$$= -\tfrac{1}{2}[0 - e^{-3}] = \frac{e^{-3}}{2}$$

$$\therefore E_0[\phi(x)] = P_0[x > \tfrac{3}{4}] = \frac{e^{-3}}{2}$$



W R O N G !

$$\cancel{\tfrac{1}{2} + \int_{3/4}^{1} e^{-2x}\,dx}$$

$$\cancel{= \tfrac{1}{2} + \tfrac{1}{2} e^{-2x}\Big|_{3/4}^{1}}$$

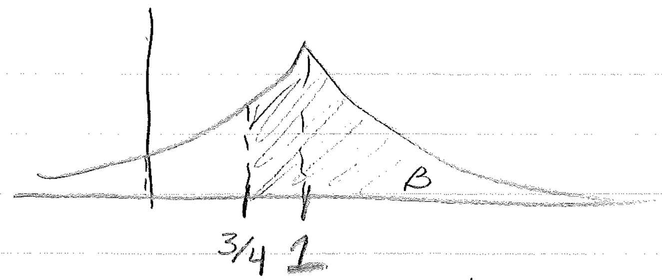
$$\cancel{= \tfrac{1}{2} - \tfrac{1}{2}(e^{-2} - e^{-3})}$$

$$\cancel{= \tfrac{1}{2}[1 - e^{-2} - e^{-3}]}$$

5

Marks

(4) (CONT)



$$\beta = E_1[\phi(x)] = P_1[X > 3/4]$$

$$\beta = \frac{1}{2} + \int_0^{1/4} e^{-2x}$$

$$= \frac{1}{2} - \frac{1}{2} e^{-2x}\Big|_0^{1/4}$$

$$= \frac{1}{2} - \frac{1}{2}\left[e^{-\frac{1}{2}} - 1\right]$$

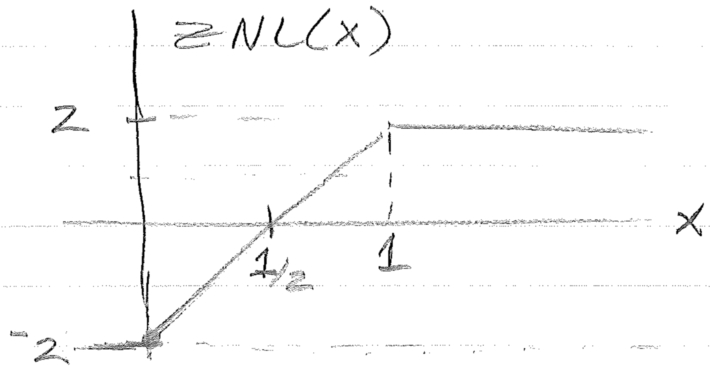
$$= \frac{1}{2} - \frac{1}{2}e^{-1/2} + \frac{1}{2}$$

$$= 1 - \frac{1}{2\sqrt{e}}$$ ✓

(CONT.)

Marks

b. N-P TEST



$f(n) = \frac{\alpha}{2} e^{-\alpha|x|}$

$\alpha = 2$

$$\phi(x) = \begin{cases} 1 & ; \ ZNL(x) > T \\ P & ; \ ZNL(x) = T \\ 0 & ; \ ZNL(x) < T \end{cases}$$

$\alpha = E_0[\phi(x)]$

$P_0[X > 1] = \int_1^\infty e^{-2x} = -\frac{1}{2} e^{-2x} \Big|_1^\infty$
$= -\frac{1}{2}[0 - e^{-2}]$
$= \frac{1}{2} e^{-2}$

$P_0[X \leq 0] = \frac{1}{2}$; $P_0[0 < X \leq 1] = 1 - \frac{1}{2} - \frac{1}{2}e^{-2}$
$= \frac{1}{2}[1 - e^{-2}]$

$\alpha = E_0[\phi(x)] = P_0[ZNL(x) > T] + P\, P_0[ZNL(x) = T]$

① α @ $T = -2$

$\alpha = P_0[ZNL(x) = -2]$
$= P_0[X < 0] = \frac{1}{2}$

② α @ $T = 2$:

$\alpha = P[ZNL(x) = 2] = P[X > 1] = \frac{1}{2} e^{-2}$

∴ Smallest α w/o randomization occurs @ $T = 2$ and
$\alpha = \frac{1}{2} e^{-2}$ ✓

⑦

Marks
1

5. $H_0 : x_k = n_k$

$H_1 : x_k = \theta s_k + n_k \; ; \; k = 1, 2, \ldots, K$

$$P_0(x_k) = \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x_k^2}{2\sigma^2}}$$

$$= (\quad)^K e^{-\sum_{k=1}^{K} x_k^2 / 2\sigma^2}$$

$$P_1(x_k) = \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_k - \theta s_k)^2}{2\sigma^2}}$$

$$= (\quad)^K e^{-\frac{\sum (x_k - \theta s_k)^2}{2\sigma^2}}$$

$$\Lambda = \frac{P_1}{P_0} = e^{-\frac{\sum (x_k - \theta s_k)^2}{2\sigma^2} + \sum x_k^2}$$

$$2\sigma^2 \ln \Lambda = -\sum (x_k - \theta s_k)^2 + \sum x_k^2$$

$$= \cancel{\sum x_k^2} - \sum \cancel{x_k^2} - 2\theta s_k x_k + \theta^2 s_k^2$$

$$= \sum 2\theta s_k x_k - \sum \theta^2 s_K^2$$

$$\sigma^2 \ln \Lambda + \frac{1}{2}\sum \theta^2 s_k^2 = \theta \sum_{k=1}^{K} s_k x_k \mathop{\gtrless}_{H_0}^{H_1} T$$

AT THIS POINT WE NEED TO KNOW SOMETHING ABOUT θ

(CONT)

2 Marks

5 (CONT)

I Case 1

Suppose $\theta > 0$,

then $\sum_{k=1}^{K} s_k x_k \underset{H_0}{\overset{H_1}{\gtrless}} \tilde{T}$

is UMP. (Can't do any better)

II Case 2

Suppose We don't know the polarity of θ. Use, then a maximum likelihood estimate. Now

$$P_1(\vec{x}_k) = (\quad)^K e^{-\frac{\sum (x_k - \theta s_k)^2}{2\sigma^2}} \quad \text{CONT} \Rightarrow$$

$$\ln P(\vec{x}_k) = \ln (\quad)^K - \frac{1}{2\sigma^2} \sum (x_k - \theta s_k)^2$$

$$= \ln (\quad)^K - \frac{1}{2\sigma^2} \sum (x_k^2 - 2\theta s_k x_k + \theta^2 s_k^2)$$

$$\frac{d}{d\theta} \ln P(\vec{x}) = -\frac{1}{2\sigma^2} \sum [-2 s_k x_k + 2\theta s_k^2]$$

$$= \frac{1}{\sigma^2} \sum [s_k x_k - \theta s_k^2]$$

$$= \frac{1}{\sigma^2} \sum s_k [x_k - \theta]$$

find $\hat{\Theta}_{ml}$

$$P(\vec{X}_K) = (\ )^K e^{\frac{\Sigma(X_K - \Theta S_K)^2}{2\sigma^2}}$$

$$\frac{d}{d\Theta} \ln P(\vec{X}) = \frac{1}{\sigma^2} \Sigma \left[S_K X_K - \Theta S_K^2 \right]$$

$$= \frac{1}{\sigma^2} \left[\Sigma S_K X_K - \Theta \Sigma S_K^2 \right]$$

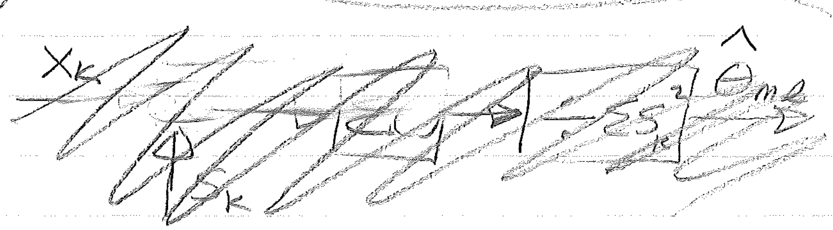
$$= \frac{1}{\sigma \Sigma S_K^2} \left[\frac{\Sigma S_K X_K}{\Sigma S_K^2} - \Theta \right]$$

$$\therefore \hat{\Theta}_{ml} = \frac{\Sigma S_K X_K}{\Sigma S_K^2}$$

$$E_\phi[\hat{\Theta}_{ml}] = \Theta \Rightarrow \text{UNBIASED}$$

$\Rightarrow \hat{\Theta}_{ml}$ IS EFFICIENT
(meets Cramer-Rao lower bound)

So, this is the best we can do
w/o knowledge of Θ's polarity $\Rightarrow$



GARBAGE

4

Marks

Recall: ~~[illegible]~~

$$\theta \sum_{K=1}^{K} s_K x_K \underset{H_0}{\overset{H_1}{\gtrless}} T$$

Substitute $\hat{\theta}_{ML}$ for θ gives

$$\frac{\left(\sum_{K=1}^{K} s_K x_K\right)^2}{\sum_{K=1}^{K} s_K^2} \gtrless T$$

This is the answer — however you skipped ~~quite a few~~ some steps

i.e., plug $\hat{\theta}_{ml}$ into $\theta \sum s_i x_i - \frac{\theta^2 \sum s_i^2}{2}$ etc.

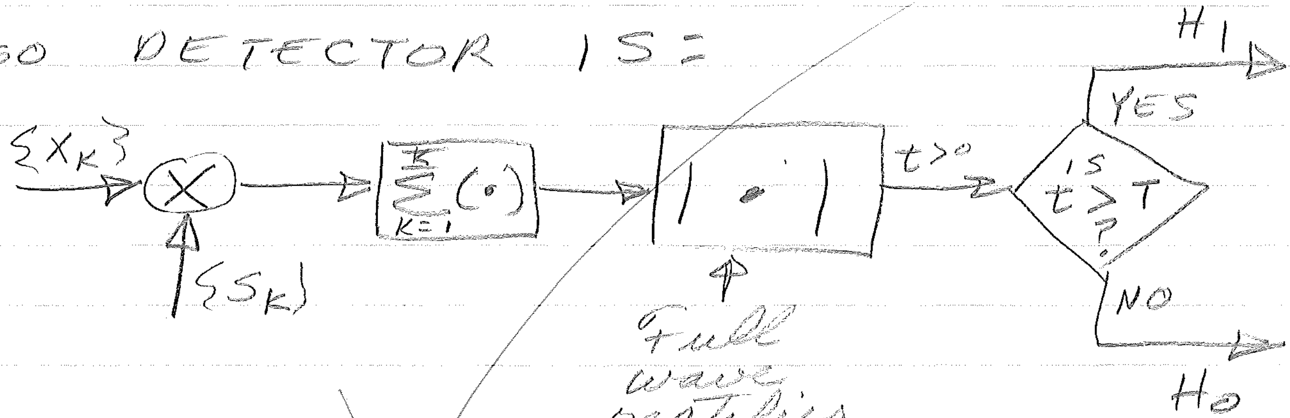
OR

$$\left(\sum_{K=1}^{K} s_K x_K\right)^2 \underset{H_0}{\overset{H_1}{\gtrless}} \tilde{T}^2$$

OR

$$\left|\sum_{K=1}^{K} s_K x_K\right| \underset{H_0}{\overset{H_1}{\gtrless}} \tilde{T} > 0$$

SO DETECTOR IS:



16

WHITED'S RULE FOR STATITISTICAL COMBINATION

GIVEN TWO STATISTICS X AND Y WITH KNOWN PROBABILITY DENSITY FUNCTIONS $p_X(x)$ AND $p_Y(y)$, AND THAT X AND Y ARE STATISTICALLY INDEPENDENT, WHITED'S LAW MAY BE APPLIED, TO FIND THE PROBABILITY DENSITY FUNCTION OF Z WHERE Z IS AN ELEMENTARY COMBINATION OF X AND Y. THAT IS

$$Z = f(X, Y) \qquad (1)$$

CONSIDER THE JOINT PDF OF Z AND Y: WE MAY WRITE

$$p_Z(z) = \int_{-\infty}^{\infty} p_{ZY}(z, y)\,dy \qquad (2)$$

BUT SINCE

$$p_{ZY}(z, y) = p_Z(z/y)\, p_Y(y) \qquad (3)$$

WE HAVE

$$p_Z(z) = \int_{-\infty}^{\infty} p_Z(z/y)\, p_Y(y)\,dy \qquad (4)$$

CONSIDER THEN, THE CUMMULATIVE DISTRIBUTION

$$P_Z(z/Y) \triangleq P_r [Z \leq z \mid Y = Y] \qquad (5)$$

FROM EQ. 1 :

$$P_Z(z/Y) = P_r [f(X, Y) \leq z \mid Y = Y]$$

$$= P_r [f(X, Y) \leq z] \qquad (6)$$

THIS IS WHITED'S RULE. FOR A GIVEN $f(X, Y)$, THIS RELATIONSIP MAY BE SOLVED, DIFFERENTIATED TO FIND $p_Z(z/Y)$, AND SUBSTITUTED INTO Eq. 4 TO FIND $p_Z(z)$

1. EXAMPLE

$$Z = f(\underline{X}, \underline{Y}) = \underline{X} + \underline{Y}$$

SUBSTITUTING INTO EQ. 6:

$$P_Z(Z/Y) = P_r[\underline{X} + Y \le Z]$$

$$= P_r[\underline{X} \le Z - Y]$$

$$= P_{\underline{X}}(Z - Y)$$

BY DEFINITION, WE HAVE

$$P_{\underline{X}}(Z - Y) = \int_{-\infty}^{Z-Y} p_x(x)\,dx$$

$$p_Z(Z/Y) = \frac{d}{dZ} P_Z(Z/Y)$$

$$= \frac{d}{dZ} P_{\underline{X}}(Z - Y)$$

USING LEIBNITZ'S RULE

$$\frac{d}{dZ} P_{\underline{X}}(Z - Y) = p_{\underline{X}}(Z - Y)\frac{\delta}{\delta Z}(Z - Y)$$
$$= p_{\underline{X}}(Z - Y) = p_Z(Z/Y)$$

SUBSTITUTING INTO Eq. 4:

$$p_Z(Z) = \int_{-\infty}^{\infty} p_{\underline{X}}(Z - Y)\, p_{\underline{Y}}(Y)\,dY$$

THUS, $p_Z(Z)$ IS THE CONVOLUTION OF THE DENSITY FUNCTIONS OF $\underline{X}$ AND $\underline{Y}$.

② EXAMPLE

$$Z = f(X, Y) = XY$$

$$Y > 0$$

INTO WHITED'S RULE:

$$P_Z(z/Y) = P_r[XY \leq z]$$

$$= P_r[X \leq \frac{z}{Y}]$$

$$= P_X(z/Y)$$

$$p_Z(z/Y) = \frac{d}{dz} P_X(z/Y)$$

$$= p_X\left(\frac{z}{Y}\right) \frac{\delta}{\delta z}\left(\frac{z}{Y}\right)$$

$$= \frac{1}{Y} p_X\left(\frac{z}{Y}\right)$$

INTO Eq. 6:

$$p_Z(z) = \int_{-\infty}^{\infty} \frac{1}{Y} p_X\left(\frac{z}{Y}\right) p_Y(Y)\, dY$$

9-3-75 (WED)

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GRADING (TENTATIVE)

- HOMEWORK - 30%, EXAMS - 40%, FINAL 30%

BIBLIOGRAPHY:

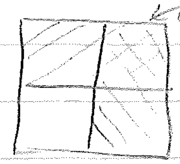
- DETECTION THEORY - I. SELINE
- TESTING STATISTICAL HYPOTHESE - LEHMANN (HEAVY)
- MATHEMATICAL STATISTICS, A DECISION THEORETIC APPROACH - FERGUSON (ONE THIRD OF COURSE)
- HELGSTRUM (TEXT)

NOTES: (HANDOUT)

TWO EVENTS A AND B ARE (STATISTICALLY) INDEPENDENT IF

$$P[AB] = P[A]P[B]$$

EX:



A

B ← INDEP., $P[AB] = P[A]P[B] = \frac{1}{4}$

IN GENERAL, N EVENTS A_i ARE MUTUALLY IND. IF

$$P[A_i A_j] = P(A_i)P(A_j)$$

$$P[A_i A_j A_k] = P[A_i)P(A_j)P(A_k)$$

$$\vdots \qquad \vdots$$

$$P(A_1 \cdots A_i A_j A_k \cdots A_N) = P(A_1) \cdots P(A_i)P(A_j)P(A_k) \cdots P(A_N)$$

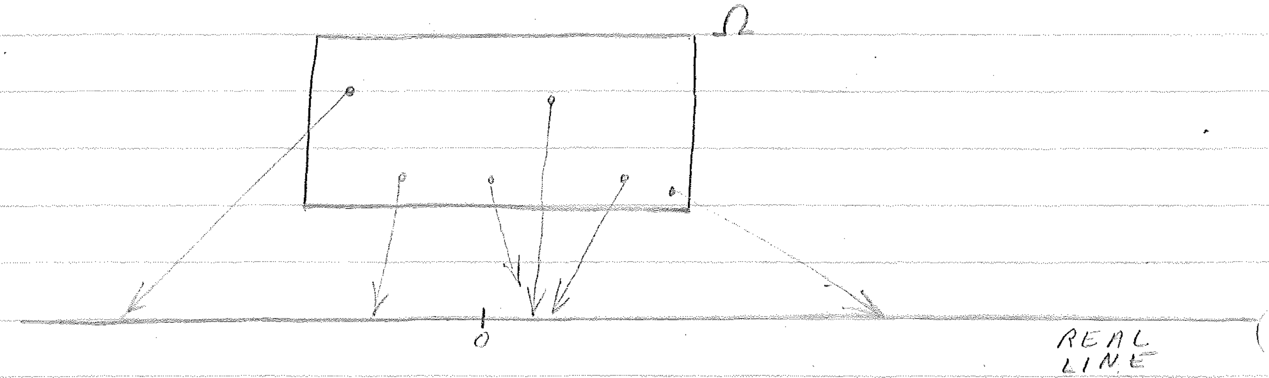
RANDOM VARIABLE

A RANDOM VARIABLE X IS A ($\mathcal{B}$-MEASUREABLE) FUNCTION FROM Ω TO THE REAL LINE.

$\mathcal{B}$-MEASURABLE: FOR ANY REAL NUMBER a, THE SET $\{\omega : X(\omega) \le a\}$ IS AN EVENT (ie BELONG TO $\mathcal{B}$)

WE ALSO REQUIRE $0 = P[X = -\infty]$; $P[X = \infty] = 1$

REPLACE $\omega \in \Omega$ WITH PTS ON THE REAL LINE.



GIVEN A REAL # x, THE SET $\{X \le x\}$ CONSISTING OF ALL OUTCOMES $\omega \in \Omega$ SUCH THAT $X(\omega) \le x$ IS AN EVENT. ITS PROBABILITY $P[X \le x]$ IS A NUMBER DEPENDING ON x; THAT IS, IT IS A FUNCTION OF x. THIS FUNCTION WILL BE DENOTED BY $F_X(x)$ AND WILL BE CALLED THE DISTRIBUTION FUNCTION OF THE RANDOM VARIABLE X. IF $F_X(x)$ HAS A DERIVATIVE, THEN WE DEFINE THE DENSITY FUNCTION $f_X(x) = \frac{d}{dx} F_X(x)$.

$f_X(x) \ge 0$ FROM THE MONOTONICITY OF $F_X(x)$.

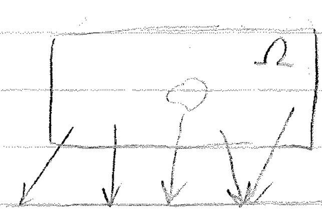
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} dF_X(x) = F_X(\infty) - F_X(-\infty) = 1$$

NOTE: ONLY TWO CONDITIONS FOR A FUNCTION TO BE A DENSITY FUNCTION

LET X BE A R.V. $f_X(x)$. LET S BE A SET OF REAL NUMBERS. THEN

$$P[X \in S) = \int_S f_X(x)\,dx$$

9-5-75 (FRI)



$\omega \in \Omega$

TIME VARYING R.V.; $X(\omega, t)$

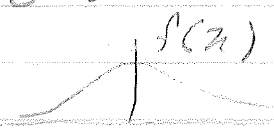
CONSIDER TESTING FOR THE PRESENCE OF A SIGNAL ON THE BASIS OF A SINGLE MEASUREMENT. Y = MEASUREMENT, S = SIGNAL, n = NOISE

H_0: $Y = n$ NULL HYPOTHESIS

H_1: $Y = s + n$

ASSUMPTIONS: (1) ASSUME THAT s IS A POSITIVE CONSTANT (2) n IS NORMALLY DISTRIBUTED WITH 0 MEAN AND VARIANCE σ^2.

$$f(n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-n^2/2\sigma^2}$$



LET $P_0(Y)$ AND $P_1(Y)$ DENOTE THE DENSITIES OF Y UNDER HYPOTHESIS H_0, AND H_1 RESPECTIVELY.

$$P_0(Y) = f(n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-Y^2/2\sigma^2}$$

$$P_1(Y) = f(n-s) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(Y-s)^2/2\sigma^2}$$

WE OBSERVE Y AND WANNA DECIDE WHETHER H_0 OR H_1 IS TRUE. SOMEHOW, WE WANNA PARTITION THE REAL LINE IN TWO DISJOINT REJIONS SUCH THAT:

IF $Y \in R_0$, THEN ANNOUNCE H_0

IF $Y \in R_1$, THEN ANNOUNCE H_1

$\ni R_1 = R_0^c$

LET Q_0 = PROBABILITY OF ANNOUGING H_1 GIVEN THAT H_0 IS TRUE

$= \int_{R_1} p_0(Y)dY$

Q_1 = PROBABILITY OF ANNOUNCHING H_0 GIVEN THAT H_1 IS TRUE

$= \int_{R_0} p_1(Y)dY$

THUS Q_0 AND Q_1 ARE THE ERROR PROBABILITES UNDER H_0 AND H_1, RESPECTIVELY. CONVENTIONALLY, $Q_0 = \alpha$ = ERROR OF THE FIRST KIND = FALSE ALARM PROBABILITY = SIZE OF THE TEST. Q_1 = ERROR OF THE SECOND KIND. CONVENTIONALLY, $\beta = 1 - Q_1$ = DETECTION PROBABILITY = POWER OF THE TEST.

NOTE: $\beta = P[\text{ANNOUNCING } H_1 / H_1 \text{ IS TRUE}]$

LET π_0 AND π_1 BE THE PRIOR PROBABILITIES OF H_0 AND H_1 RESPECTIVELY. NOTE: $\pi_1 = 1 - \pi_0$.

WE HAVE TO HAVE SOME RULE FOR CHOOSING THE DECISION REGIONS R_0 AND R_1. LET'S MINIMIZE THE PROBABILITY OF ERROR P_e. (HAVE TO KNOW PRIORS π_1 AND π_0).

$$P_e = Q_0\pi_0 + Q_1\pi_1 \qquad \text{(LAW OF TOTAL PROBABILITY)}$$

$$= \pi_0 \int_{R_1} p_0(y)dy + \pi_1 \int_{R_0} p_1(y)dy$$

$$= \pi_0 \int_{R_1} p_0(y)dy + \pi_1 \int_{R_0} p_1(y)dy + \left(\pi_0 \int_{R_0} p_0(y)dy - \pi_0 \int_{R_0} p_0(y)dy\right)$$

$$= \pi_0\left[\int_{R_1} p_0(y)dy + \int_{R_0} p_0(y)dy\right] + \int_{R_0}\left[\pi_1 p_1(y) - \pi_0 p_0(y)\right]dy$$

$$= \pi_0 + \int_{R_0}\left[\pi_1 p_1(y) - \pi_0 p_0(y)\right]dy$$

TO MINIMIZE P_e, MAKE THE INTEGRAL AS NEGATIVE AS POSSIBLE $\Rightarrow$ PUT Y IN R_0 IF $\pi_1 p_1(y) - \pi_0 p_0(y) < 0$

THEN $R_0 = \{y : \pi_1 p_1(y) - \pi_0 p_0(y) < 0\}$

ANNOUNCE H_0 IF $\dfrac{\pi_0 p_0(y)}{\pi_1 p_1(y)} > 1$

ie TO MINIMIZE P_e

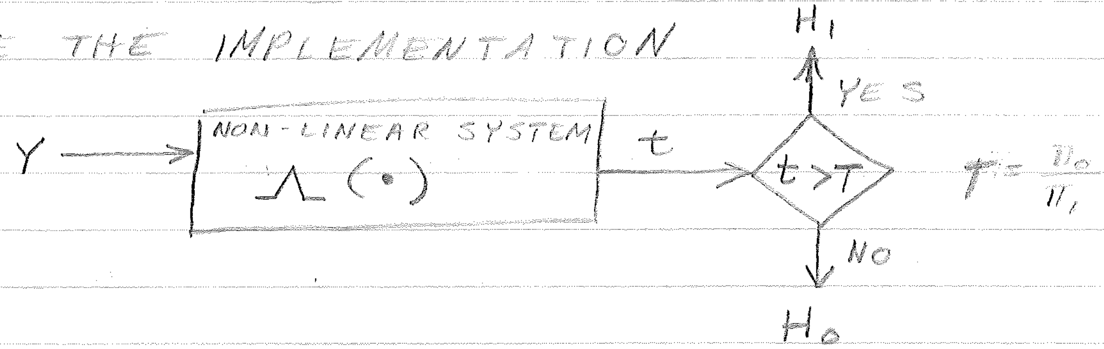
IF $\dfrac{\pi_0 p_0(y)}{\pi_1 p_1(y)} > 1$, ANNOUNCE H_0

IF $\dfrac{\pi_0 p_0(y)}{\pi_1 p_1(y)} \le 1$, ANNOUNCE H_1

DEFINE $\Lambda(Y) = \frac{P_1(Y)}{P_0(Y)}$. THIS IS CALLED THE LIKELIHOOD RATIO.

$$\Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0/\pi_1 = T$$

NOTE THE IMPLEMENTATION



OFFICE EE 152, PHONE X1258

9-8-75 (MON)

REVIEW: H_0 vs H_1

$P_0(Y)$ $\quad P_1(Y)$

$$\Lambda(Y) = P_1(Y)/P_0(Y)$$

TO MINIMIZE P_e:

$$\Lambda(Y) = \frac{P_1(Y)}{P_0(Y)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1} \quad \left(\text{IND. OF DIST. OF } P\right)$$

NOTES: EXAMPLE

$H_0: Y = n$

$H_1: Y = s + n$

THIS DETECTOR IS CALLED THE "IDEAL OBSERVER" (MIDDLETON, SIEGERT)

NOTE: $\frac{\pi_0 P_0(Y)}{\pi_1 P_1(Y)} = \frac{P(H_0/Y)}{P(H_1/Y)}$

THIS IS THE RATIO OF THE "POSTERIOR" PROBABILITIES OF THE TWO HYPOTHESIS. THUS, TO MINIMIZE THE PROBABILITY OF ERROR, WE SELECT THE HYPOTHESIS WITH THE GREATER POSTERIOR PROBABILITY.

IN OUR SPECIAL CASE (GAUSSIAN NOISE)

$$\Lambda(y) = e^{\frac{2sy - s^2}{2\sigma^2}}$$

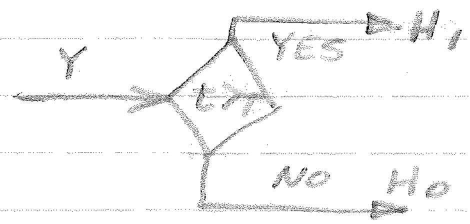
TEST IS

$$e^{\frac{2sy - s^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

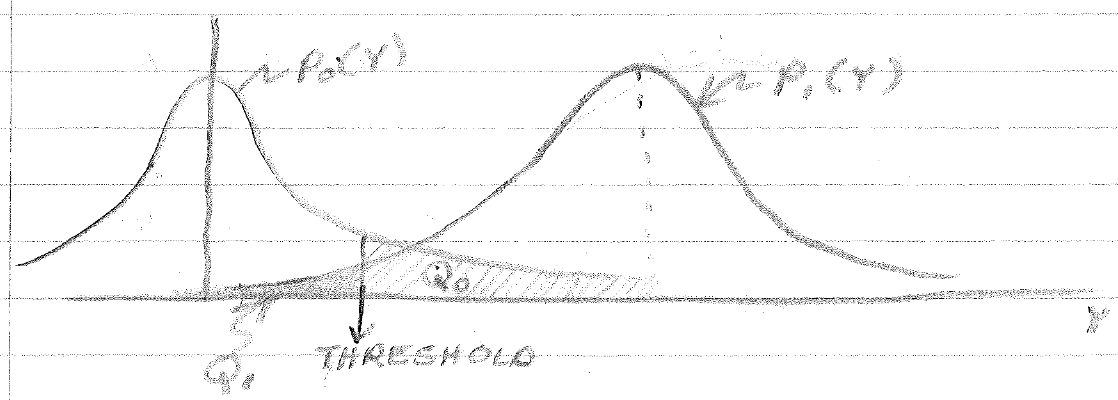
$$\frac{2sy - s^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \pi_0/\pi_1$$

$$Y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{s}{2} + \frac{\sigma^2}{s} \ln \pi_0/\pi_1 = T$$

NOTE THAT THIS IS SIMPLY A THRESHOLD DETECTOR. IN THIS CASE, WE GET RID OF THE NONLINEAR SYSTEM SIMPLY BY CHANGING THE THRESHOLD.



NOTE, FOR $\pi_0 = \pi_1$, $Y \underset{H_0}{\overset{H_1}{\gtrless}} s/2$.



WEIGHTED COST CRITERION

LET C_0 BE THE COST ASSOCIATED WITH ANNOUNCING H_1 WHEN H_0 IS TRUE. LET C_1 BE THE COST ASSOCIATED WITH ANNOUNCING H_0 WHEN H_1 IS TRUE.

ASSUME $C_1, C_2 > 0$.

LET'S MINIMIZE THE EXPECTED COST. LET J DENOTE THE EXPECTED COST

$$J = C_0 \pi_0 Q_0 + C_1 \pi_1 Q_1$$

SAME AS BEFORE (ESSENTIALLY), GIVES

$$\frac{P_1(Y)}{P_0(Y)} = \Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 C_0}{\pi_1 C_1}$$

FOR OUR SPECIAL CASE, WE GET THE FOLLOWING DETECTOR MINIMIZES THE EXPECTED COST

$$Y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{S}{2} + \frac{\sigma^2}{2} \ln \frac{C_0 \pi_0}{C_1 \pi_1}$$

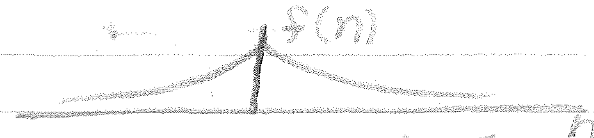
CONSIDER THE FOLLOWING PROBLEM

$H_0: Y = n$

$H_1: Y = S + n$

FOR NOISE, ASSUME LAPLACE NOISE

$$f(n) = \frac{\alpha}{2} e^{-\alpha|n|}$$



ASSUME S IS A POSITIVE CONSTANT,

THEN $p_0(Y) = \frac{\alpha}{2} e^{-\alpha|Y|}$

$p_1(Y) = \frac{\alpha}{2} e^{-\alpha|Y-S|}$

WE NOW WANNA MINIMIZE P_e

$$\Lambda(Y) = e^{-\alpha|Y-S| + \alpha|Y|}$$

$$= \begin{cases} e^{-YS} & ; Y \le 0 \\ e^{2\alpha Y - \alpha S} & ; 0 \le Y \le S \\ e^{\alpha S} & ; Y \le S \end{cases}$$

SO TEST IS

$$\Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

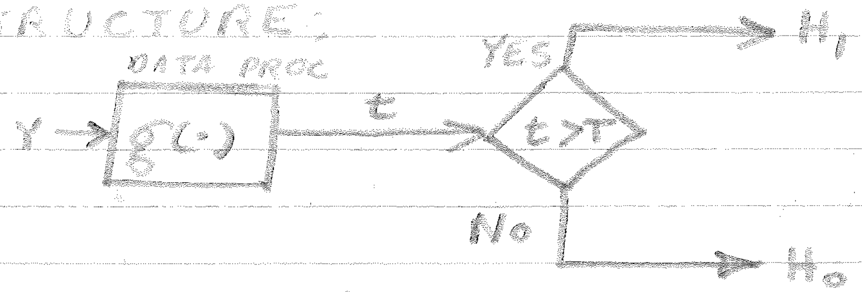
$$\ln \Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{\pi_0}{\pi_1}$$

$$\ln \Lambda(Y) = \begin{cases} -\alpha S & ; Y \le 0 \\ \alpha(2Y - S) & ; 0 \le Y \le S \\ \alpha S & ; Y \ge S \end{cases}$$

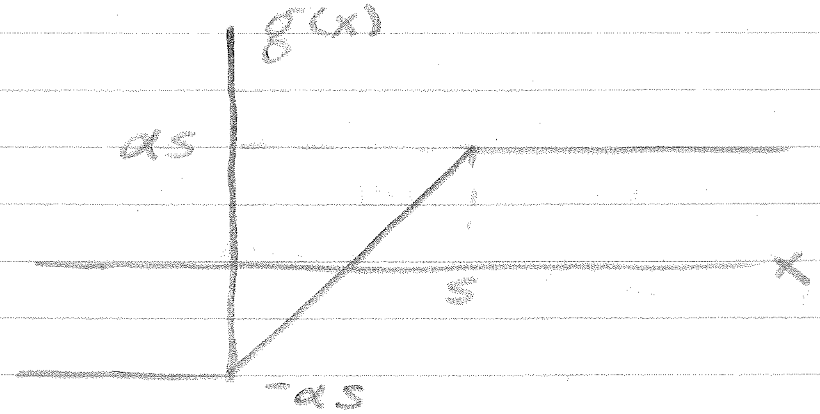
DEFINE THE FUNCTION $g(x)$

$$g(x) = \begin{cases} -\alpha S & ; x \le 0 \\ \alpha(2x - S) & ; 0 \le x \le S \\ \alpha S & ; x \ge S \end{cases}$$

THE DETECTOR HAS THE FOLLOWING STRUCTURE:



$T = \ln \frac{\pi_0}{\pi_1}$



ASSUME $\pi_0 < \pi_1 e^{-\alpha s}$

$\Rightarrow \ln \pi_0/\pi_1 < -\alpha s \leq g(Y)$

$\Rightarrow$ ALWAYS ANNOUNCE H_1

ASSUME $\pi_0 > \pi_1 e^{\alpha s}$

$\Rightarrow \ln \pi_0/\pi_1 > \alpha s \geq g(Y) \Rightarrow g(Y) < T$

$\Rightarrow$ ALWAYS ANNOUNCE H_0

9-10-75 (WED)

HOMEWORK HANDOUT DUE MON

$H_0: Y = n$

$H_1: Y = s + n$

$n \sim N(0, \sigma_n^2) \qquad s \sim N(m, \sigma_s^2)$

n AND s ARE INDEPENDENT

$$P_0(Y) = \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-Y^2/2\sigma_N^2}$$

$$P_1(Y) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_N^2+\sigma_s^2}} e^{-\frac{(Y-m)^2}{2(\sigma_N^2+\sigma_s^2)}}$$

$$\Lambda = \frac{P_1(Y)}{P_0(Y)} = \frac{\sigma_N}{(\sigma_N^2+\sigma_s^2)} \exp\left[\frac{-(Y-m)^2}{2(\sigma_N^2+\sigma_s^2)} + \frac{Y^2}{2\sigma_N^2}\right]$$

$$= \frac{\sigma_N}{\sigma_N^2+\sigma_s^2} \exp \frac{(-Y^2+2Ym-m^2)(\sigma_N)^2 + Y^2(\sigma_N^2+\sigma_s^2)}{2\sigma_N^2(\sigma_N^2+\sigma_s^2)}$$

$$\Rightarrow \Lambda(Y) = \frac{\sigma}{\sqrt{\sigma_N^2+\sigma_S^2}} \exp\left[\frac{Y^2\sigma_S^2 + 2m\sigma_N^2 Y - m^2\sigma_N^2}{2\sigma_N^2(\sigma_N^2+\sigma_S^2)}\right]$$

$$\Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1} \quad \text{GIVES}$$

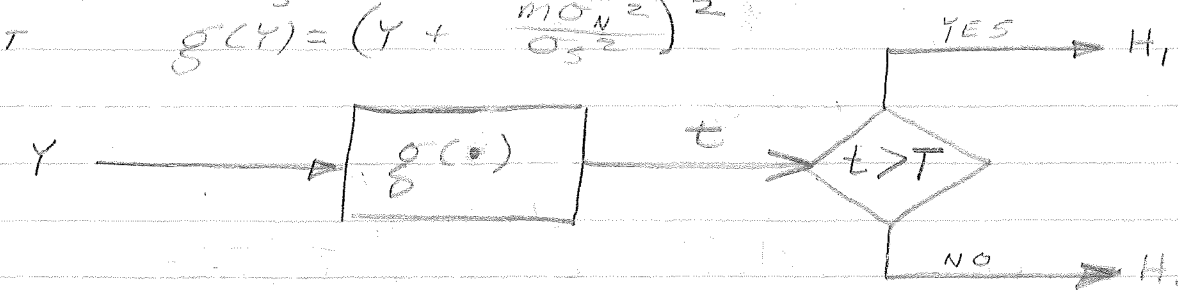
$$\frac{Y^2\sigma_S^2 + 2m\sigma_N^2 Y + m^2\sigma_N^2}{2\sigma_N^2(\sigma_N^2+\sigma_S^2)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{\pi_0\sqrt{\sigma_N^2+\sigma_S^2}}{\pi_1\sigma_N}$$

$$Y^2 + 2m\frac{\sigma_N^2}{\sigma_S^2}Y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{m^2\sigma_N^2}{\sigma_S^2} + 2\frac{\sigma_N^2}{\sigma_S^2}(\sigma_N^2+\sigma_2^2)\ln\frac{\pi\sqrt{\sigma_N^2+\sigma}}{\pi\sigma_N}$$

$$Y^2 + 2M\frac{\sigma_N^2}{\sigma_S^2}Y + \frac{m^2\sigma_N^2}{\sigma_S^4} \gtrless \frac{m\sigma_N^2}{\sigma_S^2} + 2\frac{\sigma_N^2}{\sigma_S^2}(\sigma_N^2+\sigma_S^2)\ln\frac{\pi_0\sqrt{\sigma_N^2+\sigma_S^2}}{\pi_1\sigma_N}$$

$$\left(Y + \frac{m\sigma_N^2}{\sigma_S^2}\right)^2 \underset{H_0}{\overset{H_1}{\gtrless}} T$$

LET $g(Y) = \left(Y + \frac{m\sigma_N^2}{\sigma_S^2}\right)^2$



THE GENERAL BAYES CRITERION

LET C_{ij} BE THE COST OF ANNOUNCING H_i WHEN H_j IS TRUE. ASSUME $C_{10} > C_{00}$ AND $C_{01} > C_{11}$. THE AVERAGE RISK $\mathcal{R}$ IS THE EXPECTED COST. THE BAYES CRITERION IS THE MINIMIZATION OF THE AVERAGE RISK. THE MINIMUM VALUE $\mathcal{R}$, IT DENOTED BY $\mathcal{R}_B$ AND CALLED THE BAYES RISK.

$$\mathcal{R} = \pi_0\left[C_{00}\int_{R_0} p_0(Y)\,dY + C_{10}\int_{R_1} p_0(Y)\,dY\right]$$
$$+ \pi\left[C_{01}\int_{R_0} p_1(Y)\,dY + C_{11}\int_{R_1} p_1(Y)\,dY\right]$$

WANNA MINIMIZE $\mathcal{R}$

$$
\begin{aligned}
\mathcal{R} = {} & \pi_0 C_{00} \int_{R_0} p_0(Y)\,dY + \pi_0 C_{10} \int_{R_1} p_0(Y)\,dY \\
& + \pi_1 C_{01} \int_{R_0} p_1(Y)\,dY + \pi_1 C_{11} \int_{R_1} p_1(Y)\,dY \\
& + \pi_0 C_{10} \int_{R_0} p_0(Y)\,dY - \pi_0 C_{10} \int_{R_0} p_0(Y)\,dY \\
& + \pi_1 C_{11} \int_{R_0} p_1(Y)\,dY - \pi_1 C_{11} \int_{R_0} p_1(Y)\,dY \\
= {} & \int_{R_0} (\pi_0 C_{00} - \pi_0 C_{10})\, p_0(Y)\,dY + \pi_0 C_{10} \\
& + \int_{R_0} (\pi_1 C_{01} - \pi_1 C_{11})\, p_1(Y)\,dY + \pi_1 C_{11} \\
= {} & \pi_0 C_{10} + \pi_1 C_{11} \\
& + \int_{R_0} \left[\pi (C_{01} - C_{11}) p_1(Y) - \pi_0 (C_{10} - C_{00}) p_0(Y) \right] dY
\end{aligned}
$$

$$C_{01} - C_{11} > 0 \quad ; \quad C_{10} - C_{00} > 0$$

$$\Rightarrow R_0 = \{ Y : \pi_1 (C_{01} - C_{11}) p_1(Y) - \pi_0 (C_{10} - C_{00}) p_0(Y) < 0 \}$$

$$\frac{\pi_1 (C_{01} - C_{11})\, p_1(Y)}{\pi_0 (C_{10} - C_{00})\, p_0(Y)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

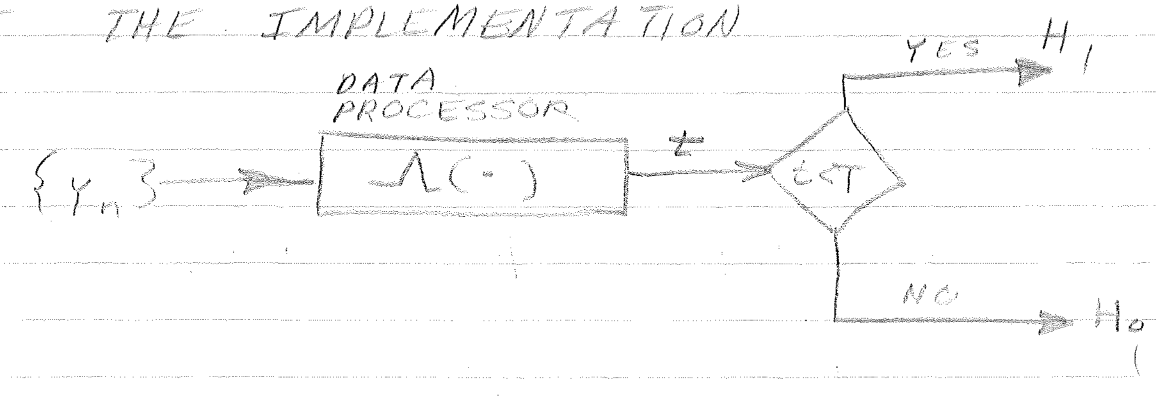
$$\Rightarrow \text{OR} \quad \Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} \Leftarrow$$

WITH k OBSERVATIONS, THE TEST

$$\Lambda(Y_1, Y_2, \ldots, Y_k) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

$$\text{WHERE} \quad \Lambda(Y_1, Y_2, \ldots, Y_k) = \frac{p_1(Y_1, Y_2, \ldots, Y_k)}{p_0(Y_1, Y_2, \ldots, Y_k)}$$

NOTE THE IMPLEMENTATION



9-12-75 (FRI)

$\mathcal{R}$ = AVERAGE RISK = $\mathcal{R}(\pi_1)$

$$\mathcal{R}(\pi_1) = \pi_0 C_{10} + \pi_1 C_{11} + \pi_1(C_{01} - C_{11})Q_1 - \pi_0(C_{10} - C_{00})(1 - Q_0) \qquad (Q_1(\pi_1), Q_0(\pi_1))$$

LET $\hat{\pi}_1$ DENOTE A PARTICULAR VALUE OF π ($\pi \subset [0,1]$). ASSUME THAT THE BAYES TEST IS DESIGNED FOR THIS PARTICULAR VALUE $\hat{\pi}_1$.

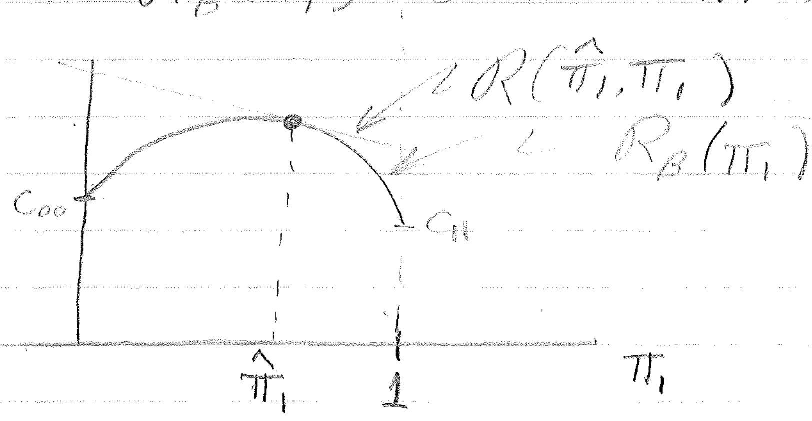
$\Longrightarrow$ R_0 AND R_1 ARE FIXED.

$\Longrightarrow$ Q_0 AND Q_1 ARE FIXED

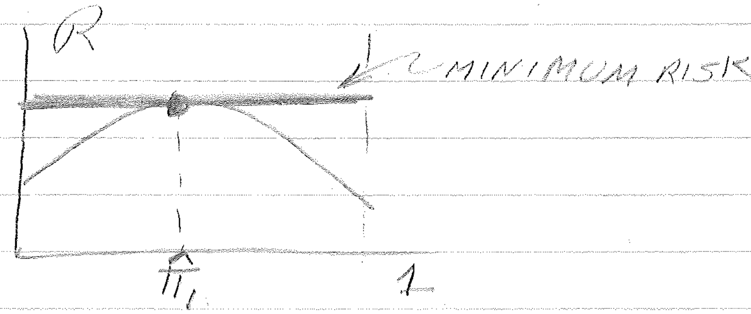
LET $\mathcal{R}(\hat{\pi}_1, \pi)$ DENOTE THE AVERAGE RISK OF THE BAYES TEST DESIGNED FOR $\hat{\pi}_1$ WHEN THE PRIOR PROBABILITY IS ACTUALLY π_1. THE GRAPH OF $\mathcal{R}(\hat{\pi}_1, \pi_1)$ VS. π_1 IS A STRAIGHT LINE. BECAUSE THIS TEST IS A BAYES TEST FOR $\pi_1 = \hat{\pi}_1$, THEN $\mathcal{R}(\hat{\pi}_1, \pi_1)$ MUST INTERSECT THE GRAPH OF $\mathcal{R}_B(\pi_1)$ AT $\pi_1 = \hat{\pi}_1$. BECAUSE THE BAYES TEST MINIMIZES THE AVERAGE RISK, WE HAVE

$$\mathcal{R}(\hat{\pi}_1, \pi_1) \geq \mathcal{R}_B(\pi_1)$$

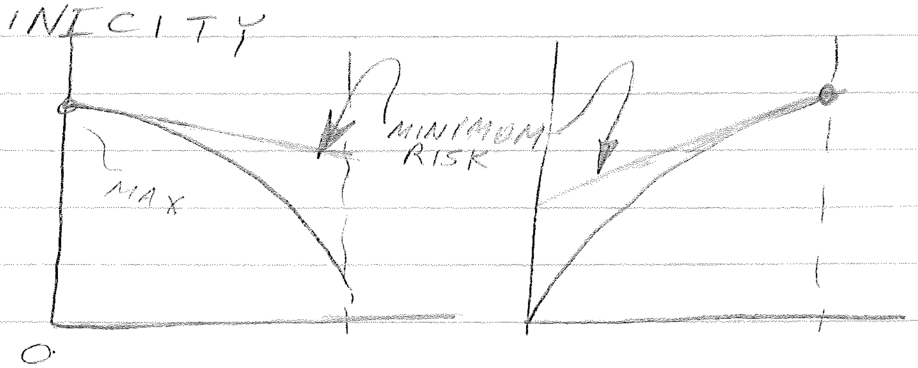
$\Rightarrow$ $-\mathcal{R}_B(\pi_1)$ IS A CONVEX FUNCTION



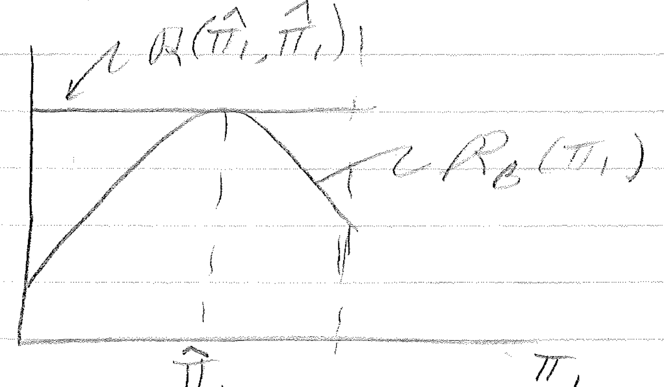
MINI-MAX CRITERION: MINIMIZE THE MAXIMUM RISK.



THE MINIMAX RISK WILL BE TANGENT TO THE BAYE'S RISK AT THE MAXIMUM VALUE OF THE BAYES RISK. FOR MONOTINICITY



IF THE MAXIMUM VALUE OF THE BAYES RISK IS INSIDE THE INTERVAL [0,1], THEN THE MINIMAX RISK WILL BE CONSTANT:



NOTE: THE CONDITION $C_{00} = C_{11} = 0$ GUARANTEES THE MAXIMUM BAYES RISK IS INSIDE THE INTERVAL [0,1]

9-15-75 (MON)

LECTURE FROM LEHMANN, Pg 65

RECALL: $\alpha = P[H_1/H_0] = Q_0$

$\beta = P[H_1/H_1] = 1 - Q_1$

α = SIZE OF THE TEST

β = POWER OF THE TEST

NEYMAN – PEARSON CRITERION IS TO CONSTRAIN α AND MAXIMIZE β. IN RADAR, THIS CORRESPONDS TO MAXIMIZING THE PROBABILITY OF DETECTING A TARGET FOR A GIVEN FALSE ALARM PROBABILITY. ie $\alpha \le \alpha_0$ (α = SIZE OF TEST, α_0 = LEVEL OF TEST) AND β = MAXIMUM.

CONSIDER A DISCRETE DISTRIBUTION.

$$\left.\begin{array}{l} H_0: \; p_n \\ H_1: \; q_n \end{array}\right\} P[n^{TH} \text{ VALUE}]$$

IF WE CONSIDER NON-RANDOMIZED TESTS, THEN THE OPTIMUM TEST IS A CRITICAL REGION S SATISFYING (S IS H_1 ACC.

$$\sum_{n \in S} p_n \le \alpha_0 \qquad \sum_{n \in S} q_n = \beta = \text{MAXIMUM}$$

THE SELECTED POINTS ARE TO HAVE A TOTAL VALUE NOT EXCEEDING α_0 ON ONE HAND AND AS LARGE AS POSSIBLE ON THE OTHER — COMPARE TO A BUYER WITH LIMITED BUDGET.

NOTE: $\phi(x)$ = THE CRITICAL FUNCTION.
= PROBABILITY OF ANNOUNCING H_1 GIVEN THE OBSERVATION IS X.

$$\alpha = P[H_1/H_0] = E_0[\phi(x)] = \int \phi(x)\, p_0(x)\, dx$$

$$\beta = P[H_1/H_1] = E_1[\phi(x)] = \int \phi(x)\, p_1(x)\, dx$$

THE NEYMANN-PEARSON LEMMA
(FROM LEHMAN, pp. 65-66)

LET P_0 AND P_1 BE PROBABILITY DISTRIBUTIONS POSSESSING DENSITIES p_0 AND p_1 RESPECTIVELY WITH RESPECT TO A MEASURE μ, THAT IS $P_0(x \in S) = \int_S p_0(x)\, d\mu(x)$. (NOTE, THIS IS WITHOUT LOSS OF GENERALITY SINCE WE CAN TAKE $\mu = P_1 + P_0$.

i. EXISTANCE: FOR TESTING $H_0 : p_0$ AGAINST THE ALTERNATIVE $H_1 : p_1$, THERE EXISTS A TEST ϕ AND A CONSTANT K SUCH THAT

(1) $E[\phi(x)] = \alpha$

(2) $\phi(x) \begin{cases} 1 & ; p_1(x) > k p_0(x) \\ 0 & ; p_1(x) < k p_0(x) \end{cases}$

ii. SUFFICIENT CONDITIONS FOR A MOST POWERFUL TEST. IF A TEST SATISFIES (1) AND (2) FOR SOME k, THEN IT IS MOST POWERFUL FOR TESTING p_0 AGAINST p_1 AT LEVEL α

iii. NESCESSARY CONDITION FOR A MOST POWERFUL TEST: IF ϕ IS MOST POWERFUL AT LEVEL α FOR TESTING p_0 AGAINST p_1, THEN FOR SOME

K IT SATISFIES (2) ALMOST EVRYWEAR, IT ALSO SATISFIES (1) UNLESS THERE EXISTS A TEST OF SIZE $<\alpha$ AND WITH POWER 1.

PROOF:

IF $\alpha = 0$, $\Rightarrow$ ANNOUNCE H_0 ANYTIME $p_0(X) > 1$

WHEN $p_0(x) = 0$, ANNOUNCE H_1

$\therefore$ PICK $K = \infty$, AND INTERPRET $0 \cdot \infty = 0$

$\alpha = 1 \Longrightarrow$ ANNOUNCE H_1 ALWAYS

PICK $K = 0$

9-17-75 (WED)

THROUGHOUT THE PROOF, ASSUME $0 < \alpha < 1$.

PART i: DEFINE $\alpha(c) \triangleq P_0[p_1(x) > c\,p_0(x)]$

$$= P_0[p_1(x)/p_0(x) > c]$$

$$1 - \alpha(c) = P_0\left[\frac{p_1(x)}{p_0(x)} \le c\right]$$

NOTE: THAT THIS IS A DISTRIBUTION FUNC.

$$1 - \alpha(-\infty) = 0 \Rightarrow \alpha(-\infty) = 1$$

$$1 - \alpha(\infty) = 1 \Rightarrow \alpha(\infty) = 0$$

$\alpha(c)$ IS NONINCREASING AND RIGHT CONTINUOUS

LET C_0 BE SUCH THAT $\alpha(C_0) \le \alpha \le \alpha(C_0 - 0)$

$$\alpha(C_0 - 0) = \lim_{t \to C_0} \alpha(t)$$

CONSIDER THE TEST DEFINED BY

$$\phi(x) = \begin{cases} 1, & p_1(x) > C_0 p_0(x) \\ \dfrac{\alpha - \alpha(C_0)}{\alpha(C_0 - 0) - \alpha(C_0)} & \text{IF } p_1(x) = C_0 p_0(x) \\ 0 & \text{IF } p_1(x) < C_0 p_0(x) \end{cases}$$

NOTE: MIDDLE EXPRESSION IS MEANINGFUL UNLESS $\alpha(C_0) = \alpha(C_0 - 0)$. IN THIS CASE $P_0[p_1(X) = C_0 p_0(X)] = 0$

$\Longrightarrow \phi$ IS DEFINED "ALMOST" EVERYWHERE

THE SIZE OF THE TEST IS

$$E_0[\phi(x)] = 1 \times P_0\left[\frac{p_1(x)}{p_0(x)} > C_0\right] + \frac{\alpha - \alpha(C_0)}{\alpha(C_0 - 0) - \alpha(C_0)} \times [\alpha(C_0 - 0) - \alpha(C_0)] + 0 \cdot P_0\left[\frac{p_1(x)}{p_0(x)} < C_0\right]$$

$$\Rightarrow E_0[\phi(x)] = \alpha(C_0) + \alpha - \alpha(C_0) = \alpha$$

THUS, C_0 CAN BE TAKEN AS THE "K" OF THE THEOREM.

— IT IS INTERESTING TO NOTE THAT C_0 IS ESSENTIALLY UNIQUE. THE ONLY EXCEPTION IS THE CASE THAT AN INTERVAL OF C'S EXISTS FOR WHICH $\alpha(c) = \alpha$. CONSIDER AN INTERVAL OF C'S FOR WHICH $\alpha(c) = \alpha$. LET (c', c'') BE SUCH AN INTERVAL. LET

$$S = \left\{x : p_0(x) > 0 \text{ AND } c' \leq \frac{p_1(x)}{p_0(x)} \leq c''\right\}$$

THEN $P_0(S) = \alpha(c') - \alpha(c'' - 0) = 0$

$p_0 = \frac{dP}{d\mu}$ — RADON-NIKODYN DERIVATIVE

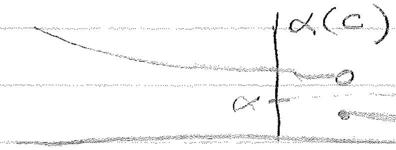
$$\Rightarrow P_0(S) = \int_S p_0 \, d\mu = 0$$

BUT $p_0(x) > 0 \Rightarrow \mu(S) = 0$

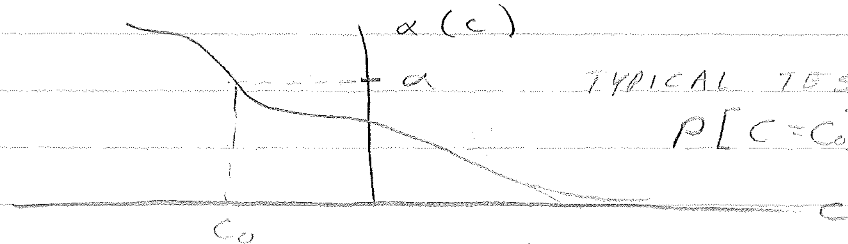
$$\therefore P_1(S) = 0 = P_0(S)$$

THUS, SETS CORRESPONDING TO TWO DIFFERENT VALUES OF C DIFFER ONLY ON A SET OF POINTS WHICH HAS PROBABILITY ZERO UNDER BOTH HYPOTHESIS. THUS, "A.E." UNIQUE.

ASIDE

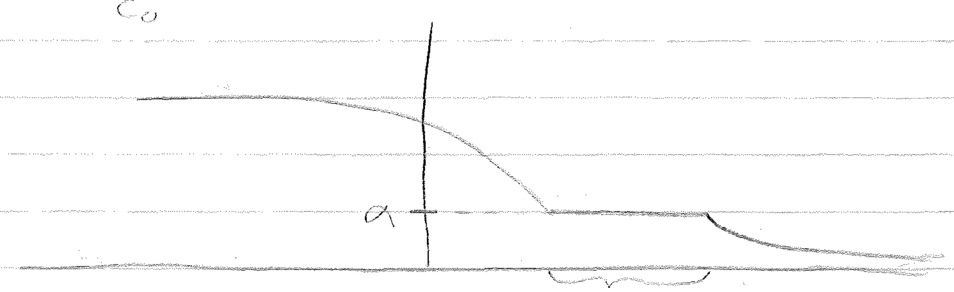


$\Rightarrow$ DIVIDE UP PROPORTIONATELY



TYPICAL TEST.

$P[C = C_0] = 0$



MATTERS NOT WHERE YOU GET C_0

PART ii) SUPPOSE THAT ϕ IS A TEST SATISFYING (1) AND (2), AND THAT ϕ^* IS ANY OTHER TEST WITH $E_0[\phi^*(x)] \leq \alpha$

LET S^+ AND S^- BE THE SETS WHERE $\phi(x) - \phi^*(x)$ > 0 AND < 0 RESPECTIVELY.

IF $x \in S^+$, $\phi(x) > 0$ AND $p_1(x) \geq k p_0(x)$

& IF $x \in S^-$, $p_1(x) \leq k p_0(x)$ (ie $\phi(x) < 1$)

$$\begin{cases} \text{IF } x \in S^+, & [\phi(x) - \phi^*(x)][p_1(x) - kp_0(x)] \geq 0 \\ \text{IF } x \in S^-, & [\phi(x) - \phi^*(x)][p_1(x) - kp_0(x)] \geq 0 \end{cases}$$

$$\Rightarrow \int [\phi - \phi^*][p_1 - kp_0]\, d\mu = \int_{S^+ \cup S^-} [\phi - \phi^*][p_1 - kp_0] \geq 0$$

THUS THE DIFFERENCE IN POWER TWIXT ϕ AND ϕ^* SATISFIES

$$\int [\phi - \phi^*] p_1 \, d\mu \geq k \int (\phi - \phi^*) p_0 \, d\mu \geq 0$$

THUS ϕ IS MORE POWERFUL THAN ϕ^*

$\Rightarrow \beta \geq \beta^*$

PART iii. LET ϕ^* BE MOST POWERFUL AT LEVEL α FOR TESTING p_0 AGAINST p_1, AND LET ϕ SATISFY (1) AND (2). LET S BE THE INTERSECTION OF THE $S^+ \cup S^-$ (i.e. ON WHICH ϕ AND ϕ^* DIFFER) WITH THE SET $\{x : p(x) \neq k\, p_0(x)\}$, AND SUPPOSE $\mu(S) > 0$. SINCE $(\phi - \phi^*)(p_1 - kp_0) > 0$ ON S, IT FOLLOWS THAT

$$\int_{S^+ \cup S^-} (\phi - \phi^*)(p_1 - k_0 p_0)\, d\mu = \int_S (\phi - \phi^*)(p_1 - kp_0)\, d\mu > 0$$

$\Rightarrow \phi$ IS MORE POWERFUL THAN ϕ^*. THE IS A CONTRADICTION, THUS $\mu(S) = 0$.

NOTE: THIS IMPLIES $P_1(S) = P_0(S) = 0$

IF ϕ^* WERE OF SIZE $< \alpha$ AND POWER < 1, IT WOULD BE POSSIBLE TO INCLUDE IN THE REJECTION REGION ADDITIONAL POINTS OR PORTIONS OF POINTS AND THEREBY TO INCREASE THE POWER UNTIL EITHER THE POWER IS 1 OR THE SIZE α

THUS, EITHER:

$$E_0[\phi^*(x)] = \alpha$$

$$\text{OR} \quad E_1[\phi^*(x)] = 1$$

Q.E.D.

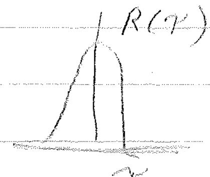
9-18-75 (FRI)

NOTE: CONSIDER THE SET ON WHICH $p_1(x) = k p_0(x)$. ON THIS SET, ϕ CAN BE DEFINED ARBITRARILY PROVIDING THAT THE RESULTING TEST HAS SIZE α. WE HAVE SHOWN THAT IT IS ALWAYS POSSIBLE TO DEFINE ϕ TO BE CONSTANT ON THIS BOUNDRY SET.

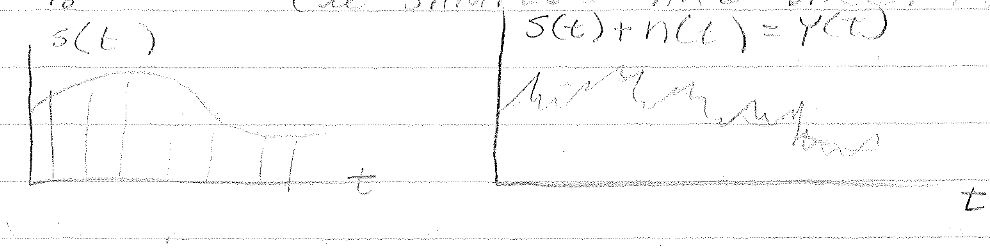
IN GENERAL, RANDOMIZATION IS NOT REQUIRED EXCEPT POSSIBLY ON THE BOUNDRY SET WHERE IT MAY BE NECESSARY TO RANDOMIZE IN ORDER TO GET THE SIZE EQUAL TO α.

IN PRACTICE, ONE WILL FREQUENTLY TO ADOPT A DIFFERENT VALUE FOR THE LEVEL OF SIGNIFICANCE WHICH DOES NOT REQUIRE RANDOMIZATION.

EXAMPLE: KNOWN SIGNAL IN WHITE GAUSSIAN NOISE NEYMAN-PEARSON TEST



← AUTOCORRELATION. FOR SAMPLES $> \tau_0$ APART, SAMPLES ARE INDEPEN. (ie SAMPLES ARE UNCORRELATED)



H_0: $Y_k = n_k$ $\quad k = 1, 2, \ldots, K$

H_1: $Y_k = s_k + n_k$

ASSUME i.i.d. (INDEPENDENT IDENTICAL DISTRIBUTION)

$n_k \sim N(0, \sigma^2)$

$$\Rightarrow p_0(Y_1, Y_2, \ldots Y_K) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^K \exp -\frac{1}{2\sigma^2}(Y_1^2 + Y_2^2 \cdots + Y_K^2)$$

$$\Rightarrow P_1(Y_1, Y_2 \ldots Y_K) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^K \exp -\frac{1}{2\sigma^2}\left[(Y_1 - S_1)^2 + (Y_2 - S_2)^2 + \ldots + (Y_K - S_K)^2\right]$$

$$\lambda(Y_1, Y_2 \ldots Y_K) = \frac{P_1(Y_1, Y_2 \ldots Y_K)}{P_0(Y_1, Y_2 \ldots Y_K)}$$

$$= \exp -\frac{1}{2\sigma^2}\left[(Y_1 - S_1)^2 - Y_1^2 + \ldots + (Y_K - S_K)^2 - Y_K^2\right]$$

WE KNOW FROM NEYMANN-PEARSON

$$\Lambda(Y_1, Y_2, Y_3, \ldots Y_4) \underset{H_2}{\overset{H_1}{\gtrless}} C$$

$$-\frac{1}{2\sigma^2}\left[-2Y_1S_1 + S_1^2 - 2Y_2S_2 + S^2 \ldots - 2Y_KS_K + S_K^2\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln C$$

$$\sum_{j=1}^{K} Y_j S_j \underset{H_0}{\overset{H_1}{\gtrless}} C_0 \Rightarrow C_0 \text{ CONTAINS } S_{i_j}^2$$

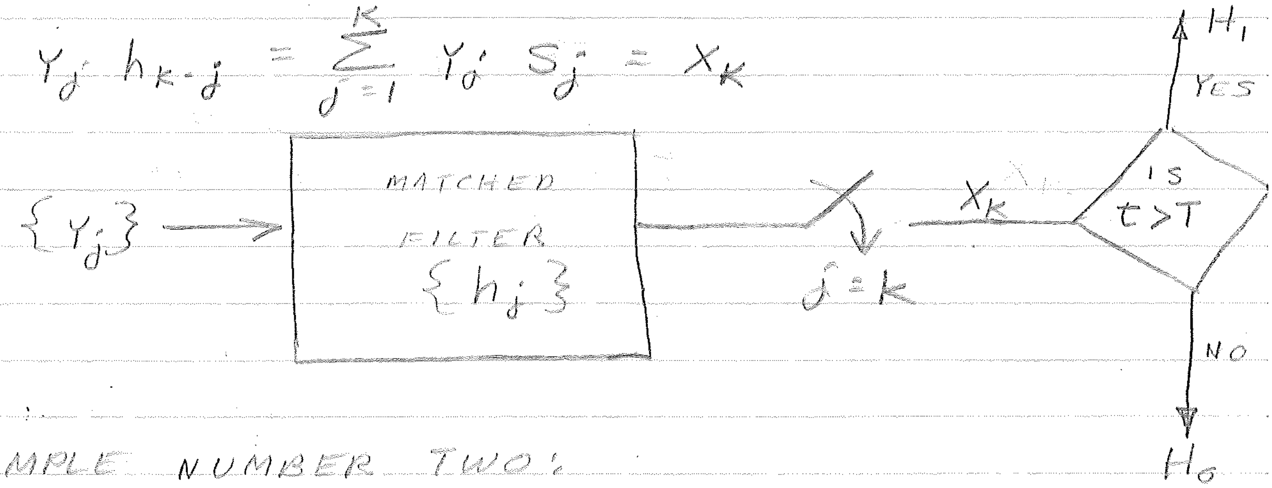
LET $X_K = \sum_{j=1}^{K} X_j S_j$

$\therefore$ THE TEST IS $X_K \underset{H_0}{\overset{H_1}{\gtrless}} C_0$

LET THE LINEAR SYSTEM (DIGITAL FILTER) BE DEFINED BY

$$h_j = \begin{cases} S_{K-j} & ; j = 0, 1, 2, \ldots, K-1 \\ 0 & ; \text{OTHERWISE} \end{cases}$$

$$\sum_{j=1}^{K} Y_j h_{K-j} = \sum_{j=1}^{K} Y_j S_j = X_K$$



EXAMPLE NUMBER TWO:

CONSIDER THE CASE OF A CONSTANT POSITIVE SIGNAL $S_j = S > 0$

$H_0: Y_K = n_K$

$H_1: Y_K = S + n_K$

THE TEST IS; $X = \sum_{j=1}^{K} Y_j \underset{H_0}{\overset{H_1}{\gtrless}} T$

UNDER H_0:

X IS GAUSSIAN

MEAN 0

VARIANCE $K\sigma^2$

UNDER H_1

X IS GAUSSIAN

MEAN KS

VARIANCE $K\sigma^2$

$$\alpha_0 = \int_T^{\infty} \frac{1}{\sqrt{2\pi K'}\,\sigma} e^{-x^2/2K\sigma^2} dx = \text{FALSE ALARM PROB.}$$

$$= 1 - \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi K'}\,\sigma} e^{-x^2/2K\sigma^2} dx$$

$Y = \frac{X}{\sqrt{K'}\,\sigma}$ GIVES

$$\alpha_0 = 1 - \int_{-\infty}^{T/\sqrt{K'}\sigma} \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dY$$

$$= 1 - \Phi\left(T/\sqrt{K'}\,\sigma\right)$$

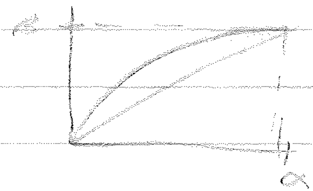
WHERE $\Phi(U) \triangleq \int_{-\infty}^{U} \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dY$

$$1 - \alpha_0 = \Phi\left(\frac{T}{\sqrt{K'}\,\sigma}\right)$$

GIVES $\boxed{T = \sqrt{K'}\,\sigma\,\Phi^{-1}(1-\alpha_0)}$

9-22-75 (MON)

RECEIVING OPERATING CHARICTERISTIC



IN BOTH PROBLEMS, PROVE A COUNTERDICTION

NOTES:

FOR CONSTANT SIGNAL IN WHITE GAUSSIAN NOISE

$$T = \sqrt{K'}\,\sigma\,\Phi^{-1}(1-\alpha_0)$$

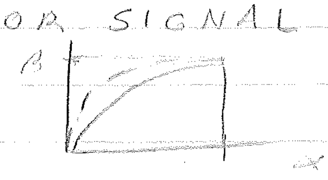
$$\beta = \int_T^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{(x-KS)^2}{2K\sigma^2}} dx$$

LET $Y = \frac{X - KS}{\sqrt{K'}\,\sigma}$

$$\Rightarrow \beta = 1 - \int_{-\infty}^{\frac{T-KS}{\sqrt{K'}\sigma}} \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dY$$

$$= 1 - \Phi\left(\frac{T-KS}{\sqrt{K'}\,\sigma}\right)$$

INCREASING SAMPLE POINTS OR SIGNAL STRENGTH IMPROVES β:



NOTE: WE CAN SET THE TRESHOLD T WITHOUT KNOWEDGE OF S (ASSUMING S IS POSITIVE). NP LEMMA GUARANTEES US THAT THE RESULTING TEST MAXIMIZES β, i.e., IT IS MOST POWERFUL WHATEVER THE VALUE OF S. NATURALLY, THE RESULTING VALUE OF β DEPENDS ON S, BUT IN ANY CASE IS MAXIMIZED.

EXAMPLE (FROM HELSTRUM pg 96) TESTING FOR WHICH OF TWO SOURCES OF NOISE IS PRESENT

$$H_0: p = p_0 \sim N(0, \sigma_0^2)$$
$$H_1: p = p_1 \sim N(0, \sigma_1^2)$$

ASSUME $\sigma_0 < \sigma_1$

WE OBSERVE K INDEPENDENT REALIZATIONS OF THE NOISE.

$$p_0(x_1, x_2, \ldots, x_K) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^K \exp -\frac{1}{2\sigma_0^2}(x_1^2 + x_2^2 + \ldots + x_K^2)$$
$$p_1(x_1, x_2, \ldots, x_K) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_1}\right)^K \exp -\frac{1}{2\sigma_1^2}(x_1^2 + x_2^2 + \ldots + x_K^2)$$

$$\Lambda(x_1, x_2, \ldots, x_K) = \left(\frac{\sigma_0}{\sigma_1}\right)^K \exp\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\sum_{j=1}^{K} x_j^2$$

THUS THE N-P TEST REDUCES TO

$$\sum_{j=1}^{K} x_j^2 \underset{H_0}{\overset{H_1}{\gtrless}} R^2 \qquad \text{(HYPERSPHERE!)}$$

NOTE: THE DECISION BOUNDRY IS A HYPERSPHERE IN K DIMENSIONAL SPACE. IF THE POINT $(x_1, x_2, \ldots x_K)$ LIES INSIDE THE HYPERSPHERE, WE ANNOUNCE H_0; OTHERWISE WE ANNOUNCE H_1.

ACCORDING TO LP LEMMA, WE CHOOSE R SUCH THAT AN ERROR OF THE FIRST KIND HAS PROBABILITY α.

$$\sum_{j=1}^{k} \frac{X_j^2}{\sigma_0^2} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{R^2}{\sigma_0^2}$$

NOTICE THAT UNDER H_0, $\frac{X_j}{\sigma_0}$ IS NORMAL WITH ZERO MEAN AND UNIT VARIANCE. DEFINE $S \triangleq \sum_{j=1}^{k} X_j^2/\sigma^2$. UNDER H_0, S HAS A CHI-SQUARED DISTRIBUTION WITH k DEGREES FREEDOM (χ_k^2).

R IS SUCH THAT

$$P_0\left[S > \frac{R^2}{\sigma_0^2}\right] = \alpha$$

$$f_0(s) = \frac{s^{\frac{k}{2}-1} e^{-s/2}}{2^{k/2}\,\Gamma(k/2)} \quad ; s \geq 0$$

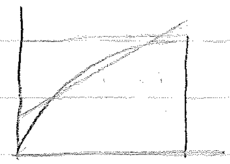
$$\int_{\frac{R^2}{\sigma_0^2}}^{\infty} f_0(s)\,ds = \alpha$$ OR USE χ^2 TABLES.

9-24-75 (WED)

HOMEWORK: 1. LOOK AT VAN-TREES 6321 RESERVE

LOOK AT LEYMANN, CHAPT. 3

#2.



MILLER & THOMAS "DETECTORS FOR DISCRETE TIME SIGNALS IN NON-GAUSSIAN NOISE."
IEEE TRANS. ON INFO. TH., MARCH, 1972 (241-250)

NOTES:

EX: SURE SIGNAL IN GAUSSIAN NOISE

$H_0: Y_k = n_k$

$H_1: Y_k = S_k + n_k \qquad k = 1, 2, 3, \ldots K$

ASSUME THAT THE SEQUENCE $n_1, n_2, \ldots, n_K$ IS MUTUALLY GAUSSIAN WITH ZERO MEANS AND COVARIANCE MAXRIX R.

$R = [r_{ij}] \ni r_{ij} = E[n_i n_j]$

ASIDE:

$n(t) \Rightarrow n_i = n(t_i)$

$R(t_1, t_2) = E[n(t_1), n(t_2)]$

$E[n_i(t), n(t_j)] = R(t_i, t_j) = r_{ij}$

ASSUME DENSITY EXISTS $\Rightarrow$ $R = [r_{ij}]$ IS INVERTABLE

LET $U = R^{-1} = [U_{ij}]$

NORMAL PDF'S:

$$P_0(Y_1, Y_2, \ldots, Y_K) = (2\pi)^{-\frac{K}{2}} \sqrt{\det(U)}\, \exp\left[-\frac{1}{2} \sum_{j=1}^{K} \sum_{k=1}^{K} U_{jk} Y_j Y_k\right]$$

$$P_1(Y_1, Y_2, \ldots, Y_K) = (2\pi)^{-K/2} \sqrt{\det(U)}\, \exp\left[-\frac{1}{2} \sum_{j=1}^{K} \sum_{k=1}^{K} U_{jk} (Y_j - S_j)(Y_k - S_k)\right]$$

(CONT)

$$\Lambda(Y_1 Y_2,\ldots,Y_K) = \exp -\frac{1}{2}\sum_{j=1}^{K}\sum_{k=1}^{K} U_{jk}\left[(Y_j - S_j)(Y_k - S_k) - Y_j Y_k\right]$$

$Y_j S_k = Y_k S_j$ DUE TO SYMMETRY OF COVARIANCE MATRIX

$$= \exp \frac{1}{2}\sum_{j=1}^{K}\sum_{k=1}^{K} U_{jk}\left[Y_j S_k + Y_k S_j - S_j S_k\right]$$

$$= \exp \sum_{j=1}^{K}\sum_{k=1}^{K} U_{jk}\left[Y_j S_k - \frac{1}{2} S_j S_k\right]$$

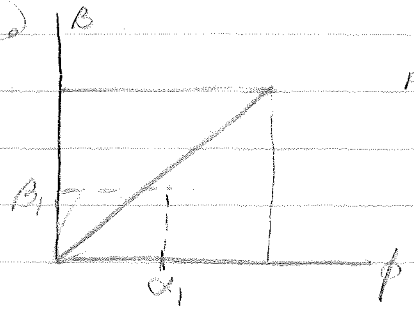
THE N.P. TEST IS

$$\sum_{j=1}^{K}\sum_{k=1}^{K} U_{jk} Y_j S_k \underset{H_0}{\overset{H_1}{\gtrless}} T$$

9-26-75 (FRI)

HOMEWORK PROBLEM:

(1)



ASSUME $\beta_1 < \alpha_1$

N.P.

$\Lambda > \Lambda_0 \Rightarrow H_1$

$\Lambda = \Lambda_0 \Rightarrow H_1$ WITH p_0

$\Lambda < \Lambda_0 \Rightarrow H_0$

$\phi(x) = P$[ANNOUNCING H_1 WHEN OBSERVATION IS x]

$\alpha = \int \phi(x) P_0(x)\,dx = E_0[\phi(x)]$

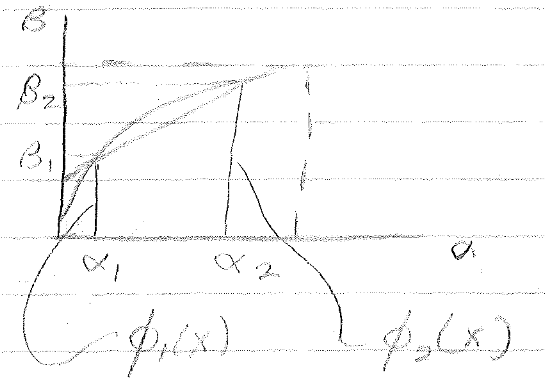
$\beta = \int \phi(x) P_1(x)\,dx = E_1[\phi(x)]$

FOR US, WE WANNA

$\alpha_1 = E_0\{\phi(x)\}$ $E_1\{\phi(x)\} \geq \alpha_1$

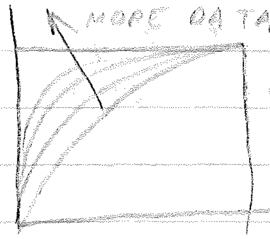
SO LET $\phi(x) = \alpha_1$.

(2)



$$\text{LET } \phi_0(x) = \frac{\alpha - \alpha_2}{\alpha_1 - \alpha_2}\,\phi_1(x) + \left(1 - \frac{\alpha - \alpha_2}{\alpha_1 - \alpha_2}\right)\phi_2(x)$$

NOW, GENERALLY



EXAMPLE: TESTING FOR A WEIGHTED DIE

LET $H_0: p_0 = 1/6 \quad , n = 1 \ldots 6,$

$H_1: P_1 = 0.2, \; p_2 = 0.16. \quad n = 2, \ldots 6$

ASSUME WE HAVE ONE OBSERVATION.

$$\Lambda(n) = \begin{cases} 1.2 & n = 1 \\ 0.96 & n \geq 0 \end{cases}$$

$P_0[\Lambda = 1.2] = 1/6$

$P_0[\Lambda = 0.96] = 5/6$

LET $\alpha = 0.05$

$$\phi(n) = \begin{cases} 1 & \text{if } \Lambda < \Lambda_0 \\ P & \text{IF } \Lambda = \Lambda_0 \\ 0 & \text{IF } \Lambda < \Lambda_0 \end{cases}$$

$\alpha = E_0[\phi(N)]$, PICK $\Lambda_0 = 1.2$

$\alpha = (1)(0) + P(\frac{1}{6}) + 0(5/6) \Rightarrow P = 0.3$

$\beta = E_1[\phi(N)] = P_1(0.2) = 0.06$

TEST IS: IF $n = 1$, ANNOUNCE H_1 WITH PROBABILITY 0.3. IF $\alpha >$ ANNOUNCE H_0

CONSIDER SAME, BUT WITH 2 INDP OBSERVATIONS

$$\Lambda(n_1,n_2)=\begin{cases}1.44 & ;\ n_1=n_2=1\\ 1.152 & ;\ n_1=1,\ n>1 \\ & \text{OR } n_1>1,\ n_2=1\\ 0.9216 & n_1>1 \text{ AND } n_2>1\end{cases}$$

$P_0[\Lambda=1.44]=1/36$

$P_0[\Lambda=1.52]=5/18$

$P[\Lambda=0.9216]=25/36$

SAY $\alpha=0.05$

$\alpha=0.05=1\cdot\frac{1}{36}+P\,\frac{5}{18}+0 \Rightarrow P=0.08$

$\beta=E_1[\phi(N_1,N_2)]=1\cdot(0.04)+P(0.32)$

$=1\cdot(0.04)+(0.08)(0.32)=0.0656$

IF BOTH OBSERVATIONS ARE $1 \Rightarrow H_1$

IF ONE OF THE " IS $1 \Rightarrow H_1$ WITH $P=0.16$

9-29-75 (MON)

$H_0: p_n$

$H_1: q_n \qquad \Lambda(n)=\frac{q_n}{p_n}$

$$\phi(n)=\begin{cases}1 & ;\ \Lambda(n)>\Lambda_0\\ p & ;\ \Lambda(n)=\Lambda_0\\ 0 & ;\ \Lambda(n)<\Lambda_0\end{cases}$$

$\Lambda(n_1)\geq\Lambda(n_2)\geq\Lambda(n_3)\geq\ldots$

$\frac{q_{n_1}}{p_{n_1}}\geq\frac{q_{n_2}}{p_{n_2}}\geq\frac{q_{n_3}}{p_{n_3}}\geq \qquad \alpha=E[\phi(x)]$

$\sum_{i=1}^{j} p_{ni}\leq\alpha$

$\sum_{i=1}^{j} p_{ni}+p\,p_{j+1}=\alpha$

IN CONTINUOUS CASE:

$\alpha=\int_T^{\infty} p_0(\Lambda)\,d\Lambda$

DETECTION OF SURE SIGNALS IN ADDITIVE WHITE NOISE

$H_0: Y_i = n_i$

$H_1: Y_i = S_i + n_i \qquad i = 1, 2, \ldots, K$

ASSUME n_i ARE INDEPENDENT REALIZATIONS
DENSITIES $f_i(\cdot)$.

N·P CRITERION

$\left\{\begin{array}{l}\text{COMPUTE } \Lambda(Y_1, Y_2, \ldots, Y_K) \text{ AND COMPARE} \\ \text{WITH A THRESHOLD. THE VALUE OF} \\ \alpha \text{ DETERMINES THE THRESHOLD.}\end{array}\right.$

$$\Lambda(Y_1, Y_2, \ldots, Y_K) = \frac{P_1(Y_1, Y_2, \ldots, Y_K)}{P_0(Y_1, Y_2, \ldots, Y_K)}$$

BECAUSE OF STATISTICAL INDEPENDENCE

$$P_0(Y_1, Y_2 \ldots Y_K) = \prod_{i=1}^{K} f_i(Y_i)$$

$$P_1(Y_1, Y_2, \ldots, Y_K) = \prod_{i=1}^{K} f_i(Y_i - S_i)$$

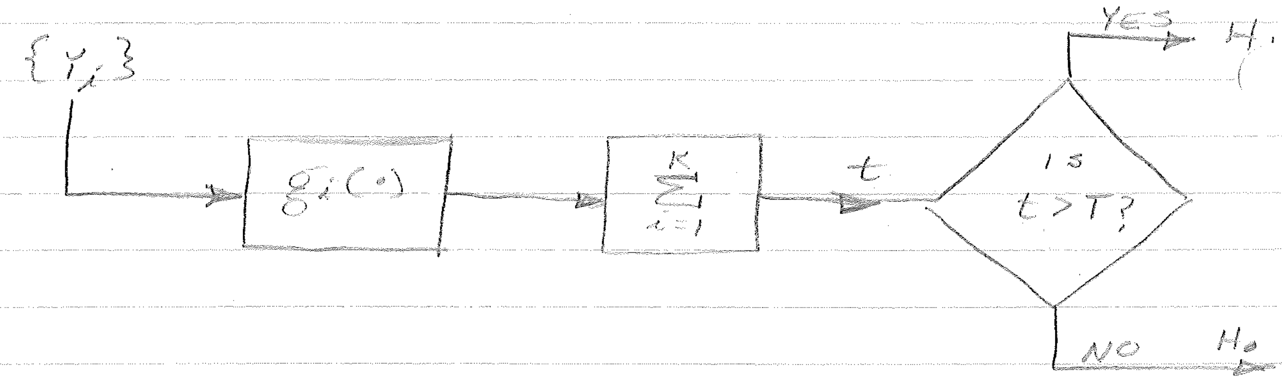
$$\Lambda(Y_1, Y_2, \ldots, Y_K) = \prod_{i=1}^{K} \frac{f_i(Y_i - S_i)}{f_i(Y_i)}$$

$$\ln \Lambda(Y_1, Y_2, \ldots, Y_K) = \sum_{i=1}^{K} \ln\left[f_i(Y_i - S_i) / f_i(Y_i)\right] \underset{H_0}{\overset{H_1}{\gtrless}} T$$

LET $g_i(Y) \triangleq \ln f_i(Y_i - S_i) / f_i(Y_i)$

MAKES TEST INTO $\displaystyle\sum_{i=1}^{K} g_i(Y_i) \underset{H_0}{\overset{H_1}{\gtrless}} T$

(CONT)

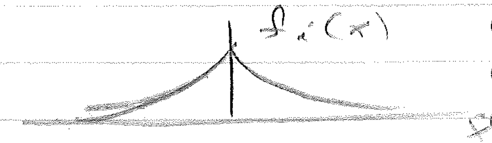


NOTE: $g_i(\cdot)$ IS A TIME-VARYING ZERO MEMORY NONLINEARITY.

FOR A SURE SIGNAL IN WHITE NOISE, THE OPTIMAL SYSTEM IS A TIME-VARYING ZERO-MEMORY NONLINEARITY FOLLOWED BY A SUMMER & A THRESHOLD DEVICE.

EXAMPLE: A POSITIVE SURE SIGNAL IN WHITE LAPLACE NOISE.

$$f_i(x) = \frac{\alpha_i}{2} e^{-\alpha_i |x|}$$



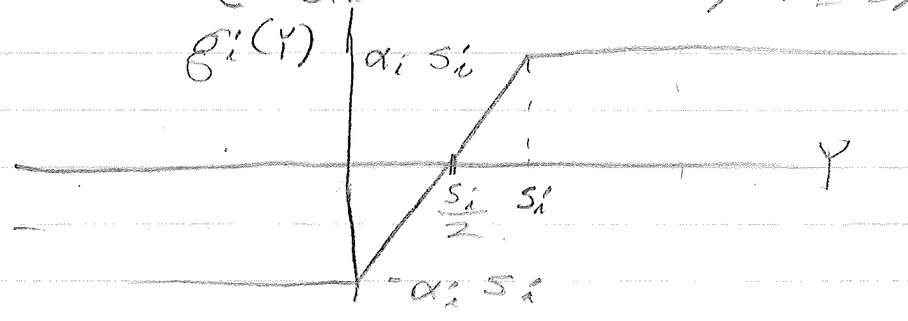
ASSUME $s_i > 0$

$$g_i(Y) = \ln \frac{f_i(Y - s_i)}{f_i(Y)}$$

$$= \ln e^{-\alpha_i |Y - s_i| + \alpha_i |Y|}$$

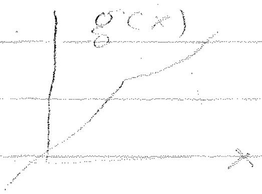
$$= -\alpha_i |Y - s_i| + \alpha_i |Y|$$

$$g_i(Y_i) = \begin{cases} -\alpha_i s_i & ; Y \leq 0 \\ 2\alpha_i Y - \alpha_i s_i & ; 0 \leq Y \leq s_i \\ \alpha_i s_i & ; Y \geq s_i \end{cases}$$



10-1-75 (WED)

1. LET $Y = g(X)$ AND ASSUME g IS MONOTONICALLY INCREASING,



THEN $X = g^{-1}(Y)$

DISTRIBUTION OF X: $F_X(Y) = P[X \le Y]$

$$= P[g^{-1}(Y) \le Y]$$
$$= P[Y \le g(Y)]$$
$$= F_Y[g(Y)]$$

DENSITY DUNCTION

$$f_X(Y) = f_Y[g(Y)] \frac{d}{dY} g(Y)$$

REPLACE Y WITH $g^{-1}(Y)$

$$f_X(g^{-1}(Y)) = f_Y(Y) \cdot g'[g^{-1}(Y)]$$

$$= \frac{f_X[g^{-1}(Y)]}{g'[g^{-1}(Y)]}$$

2. LET $Y = g(X)$ AND g IS MONOTONICALLY DECR.

$F_X(Y) = P[X \le Y] = P[X < Y]$ i.e. NO DELTAS

$$= P[g^{-1}(Y) \le Y]$$

DUE TO MONO-DECR:

$$F_X(Y) = P[Y > g(Y)]$$
$$= 1 - P[Y \le g(Y)]$$
$$= 1 - F_Y[g(Y)]$$
$$f_X(Y) = -f_Y[g(Y)] \cdot \frac{d}{dY} g(Y)$$

REPLACE Y WITH $g^{-1}(Y)$

$$\Longrightarrow f_X[g^{-1}(Y)] = f_Y(Y) \cdot g'[g(Y)]$$

$$f_Y(Y) = \frac{f_X[g'(Y)]}{-g'[g'(Y)]}$$

NOTE: $g' < 0$

MAY COMBINE TO STATE

LET $Y = g(X)$. IF g IS MONOTONIC

$$f_Y(y) = \frac{f_X[g^{-1}(y)]}{|g'\{g^{-1}(y)\}|}$$

EXAMPLE: LET $Y = g(X)$

$$g(x) = e^x - 1 = y$$

$$g^{-1}(y) = \ln(y+1)$$

$$g'(x) = e^x$$

$$g'[g^{-1}(y)] = y + 1$$

$$f_Y(y) = \frac{f_X[\ln(y+1)]}{y+1} \quad ; \; y \geq -1$$

ASSUME, FOR ILLUSTRATIVE PURPOSES, f_X IS GAUSSIAN:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

THEN

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma(y+1)} e^{-\frac{[\ln(y+1)]^2}{2\sigma^2}} \quad ; \; y > -1$$

CONSIDER THE TRANSFORMATION $\underline{Y} = \underline{X}^2$

ASSUME $Y > 0$

$$P[\underline{Y} \le y] = P[\underline{X}^2 \le y] = P[-\sqrt{y} \le \underline{X} \le \sqrt{y}]$$

$$= \int_{\sqrt{y}}^{\sqrt{y}} f_{\underline{X}}(x)\,dx$$

$$f_{\underline{Y}}(y) = \frac{d}{dx}\int_{-\sqrt{y}}^{\sqrt{y}} f_{x}(x)\,dx$$

$$= \frac{1}{2\sqrt{y}}\left[f(\sqrt{y}) + f(-\sqrt{y})\right]$$

FOR $f_{\underline{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{x^2/2\sigma^2}$

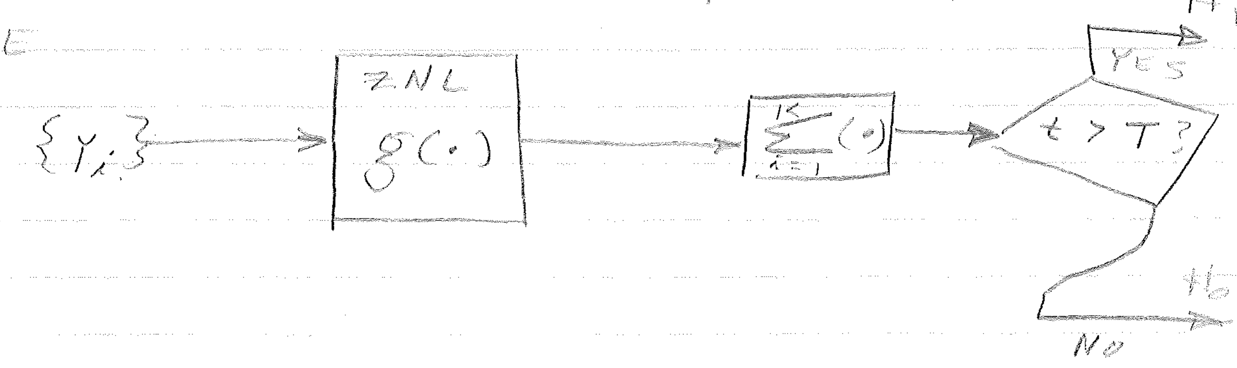
$$\Rightarrow f_{\underline{Y}}(y) = \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{y}} e^{-y^2/2\sigma^2}, \quad Y > 0$$

$\sim \chi^2$ DISTRIBUTION WITH 1 D.O.F.

BACK TO DETECTION OF SURE SIGNAL IN WHITE NOISE

$$g_i(y) = \ln\left[\frac{f_i(y - s_i)}{f_i(y)}\right]$$

THE ZERO-MEMORY NON-LINEARITY (ZNL) WILL BE FIXED IF THE SIGNAL IS CONSTANT & THE NOISE IS IDENTICALLY DISTRIBUTED, OPTIMAL DETECTOR FOR CONSTANT SIGNAL IS STATIONARY WHITE NOISE



EX: DETECTION OF POSITIVE D.C. SIGNAL IN STATIONARY WHITE LAPLACE NOISE.

$H_0: Y_i = n_i$

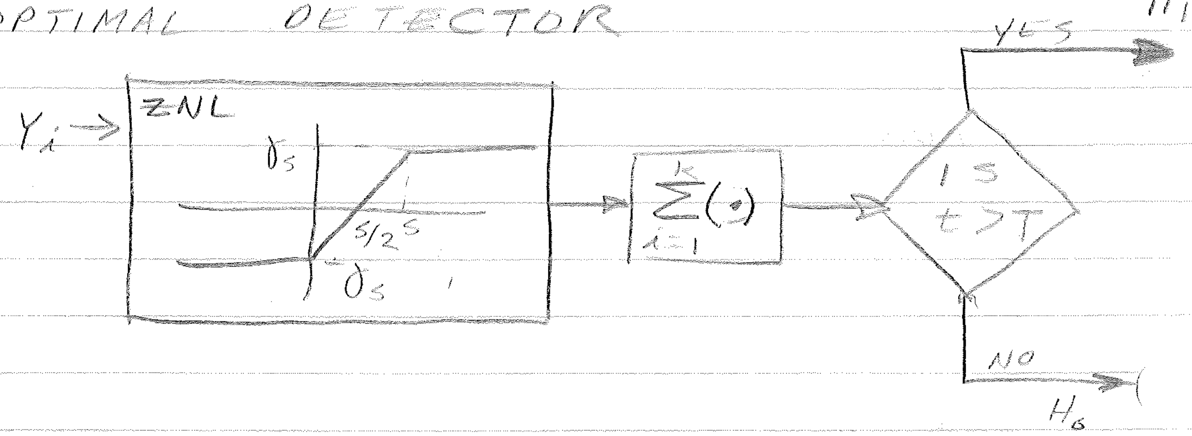
$H_1: Y_i = S + n_i$

$i = 1, 2, \ldots, K$

DENSITY OF NOISE $= \frac{\delta}{2} e^{-\delta |n_i|}$

AND $S > 0$

OPTIMAL DETECTOR



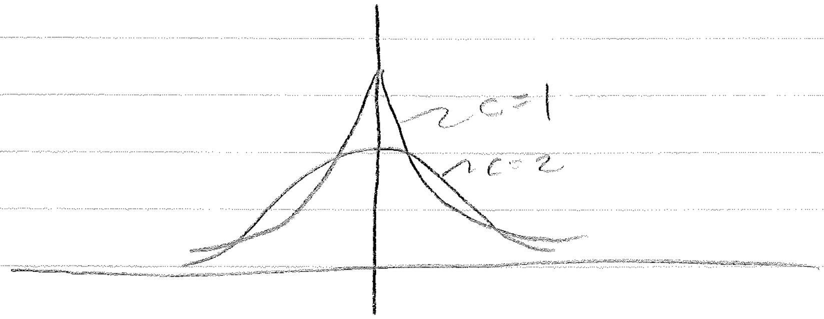
10-3-75 (FRI.) FROM MILLER-THOMAS PAPER

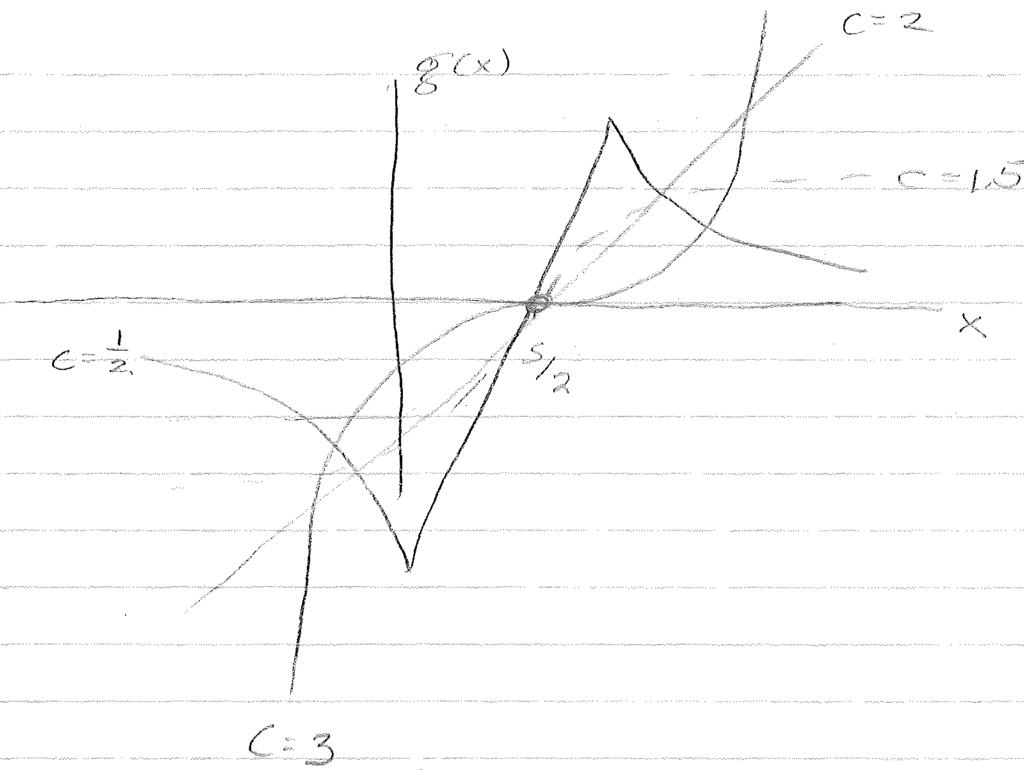
"GENERALIZED" GAUSSIAN NOISE

$f(x) = K e^{-K_1 |x|^c}$; $c > 0$

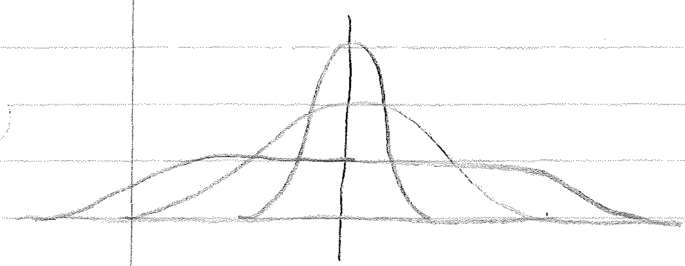
FOR $C = 2$, WE GET GAUSSIAN NOISE

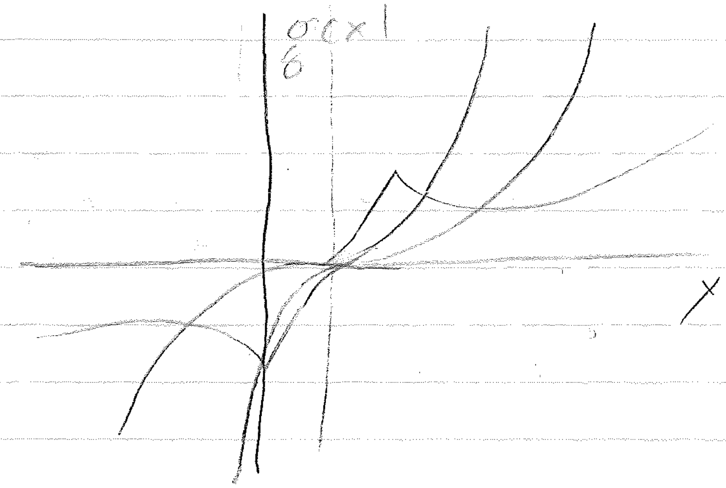
FOR $C = 1$, " " LAPLACE NOISE





GENERALIZED β NOISE

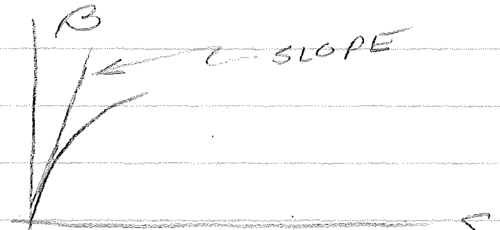




MUST PRETTY WELL KNOW THE NOISE DISTRIBUTION EXACTLY.

LOCALLY OPTIMAL TESTS (FERGUSON, pg 235)

THE LOCALLY OPTIMAL SCHEME IS A SMALL SIGNAL SCHEME.
RECALL: N-P. MAXIMIZES β FOR A GIVEN α. LET $S>0$ DENOTE THE SIGNAL. NOTICE THAT β IS A FUNCTION OF S. THE LOCALLY OPTIMAL SCHEME MAXIMIZES THE SLOPE OF β AS $S \to 0$.



THE LOCALLY OPTIMAL TEST MAXIMIZES $\frac{d}{ds} E_1[\phi(x)]\big|_{s\to 0}$ FOR A FIXED α.

MAXIMIZE

$$\frac{d}{ds}\int \phi(x)\, p_1(x,s)\, dx\Big|_{s\to 0}$$

ASSUME THAT WE CAN DIFFERENTIATE INSIDE THE INTEGRAL, WE THEN WANNA MAXIMIZE

$$\int \phi(x)\, \frac{\delta}{\delta s} p_1(x,s)\, dx\Big|_{s\to 0} = \frac{\delta \beta}{\delta s}$$

NOTE THAT WE HAVE THE GENERAL FORM OF A N-P TEST. ie

$$E_0[\phi(x)] \le \alpha$$

$$\frac{\frac{\delta}{\delta s} p_1(x,s)\big|_{s\to 0}}{p_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

CONSIDER THE CASE WHERE WE'RE TESTING FOR THE PRESENCE OF A SIGNAL IN STATIONARY WHITE NOISE. OUR TEST IS

$$\frac{\frac{\delta}{\delta s} p_1(x,s)\Big|_{s=0^+}}{p_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

10-6-75 (MON)

TEST ON MONDAY (10-13-75)

LOCALLY OPTIMUM TESTS:

OUR TEST WAS:

$$\frac{\frac{\delta}{\delta s} P_1(X,S)\Big|_{S=0^+}}{P_0(X)} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

CONSIDER THE NUMERATOR

$$\frac{\delta}{\delta s} P_1(x,s) = \frac{\delta}{\delta s} \prod_{i=1}^{K} f(x_i - s)$$

$$= \sum_{i=1}^{K} \frac{\delta}{\delta s} f(x_i - s) \prod_{\substack{j=1 \\ \neq i}}^{K} f(x_j - s)$$

$$= \sum_{i=1}^{K} \frac{\frac{\delta}{\delta s} f(x_i - s)}{f(x_i - s)} \cdot f(x_i - s) \prod_{\substack{j=1 \\ \neq i}}^{K} f(x_j - s)$$

$$= \sum_{i=1}^{K} \frac{\frac{\delta}{\delta s} f(x_i - s)}{f(x_i - s)} \prod_{j=1}^{K} f(x_j - s)$$

$$\Rightarrow \frac{\delta}{\delta s} P_1(x,s) = \prod_{j=1}^{K} f(x_j - s) \sum_{i=1}^{K} \frac{\frac{\delta}{\delta s} f(x_i - s)}{f(x_i - s)}$$

$$\frac{\delta}{\delta s} P_1(x,s)\Big|_{s=0^+} = \prod_{j=1}^{K} f(x_j) \sum_{i=1}^{K} \frac{-\frac{d}{dx_i} f(x_i)}{f(x_i)}$$

NOW $\quad p_0(x) = \prod_{j=1}^{K} f(x_j)$

$$\Rightarrow \frac{\frac{\delta}{\delta s} P_1(x,s)\Big|_{s=0^+}}{P_0(x)} = \frac{\prod_{j=1}^{K} f(x_j) \sum_{i=1}^{K} \frac{-\frac{d}{dx_i} f(x_i)}{f(x_i)}}{\prod_{j=1}^{K} f(x_j)}$$

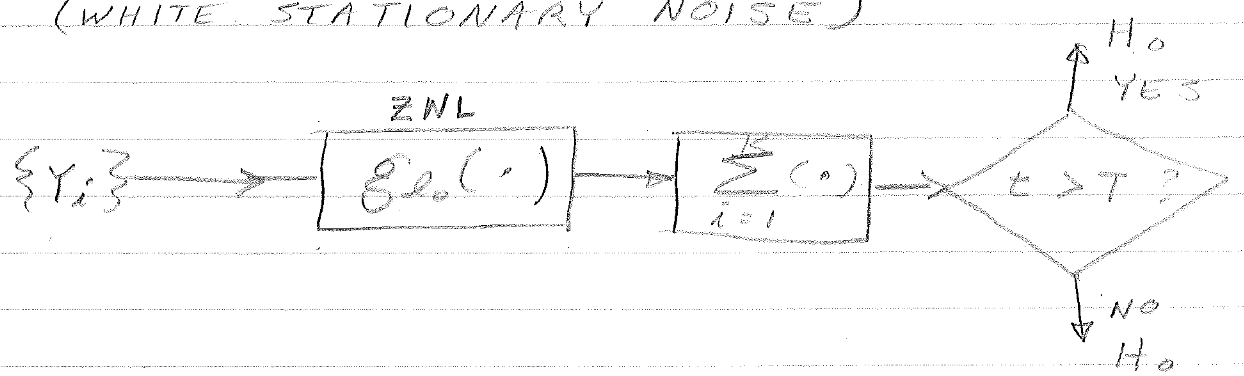
$$= \sum_{i=1}^{K} \frac{-\frac{d}{dx_i} f(x_i)}{f(x_i)}$$

$$= \sum_{i=1}^{K} -\frac{d}{dx_i} \ln\left[f(x_i)\right]$$

DEFINE $\quad g_{lo}(x) \triangleq -\frac{d}{dx} \ln f(x)$

THIS IS THE LOCALLY OPTIMAL ZERO MEMORY NONLINEARITY.

IMPLEMENTATION OF THE LOCALLY OPTIMAL N-P DETECTOR (WHITE STATIONARY NOISE)



T IS CHOSEN SUCH THAT THE FALSE ALARM PROBABILITY IS α

GIVEN $g_{lo}(x)$, WHAT IS NOISE? SOLVE

$$g_{lo}(x) = \frac{-\frac{d}{dx} f(x)}{f(x)}$$

$$\Rightarrow \frac{d}{dx} f(x) + g_{lo}(x) f(x) = 0$$

SOLVING THIS D.E., WE GET

$$f(x) = K \exp\left[-\int_0^x g_l(y)\,dy\right]$$

K IS THE NORMALIZING CONSTANT

THIS IS THE NOISE DENSITY FOR WHICH $g_{lo}(x)$ IS THE LOCALLY OPTIMAL NON-LINEARITY.

EXAMPLES: LOCALLY OPTIMAL DETECTORS FOR STATIONARY WHITE NOISE.

GAUSSIAN NOISE: $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$

$$\ln f(x) = \ln \frac{1}{\sqrt{2\pi}\,\sigma} - \frac{x^2}{2\sigma^2}$$

$$-\frac{d}{dx} \ln f(x) = x/\sigma^2$$

NOTE: THIS IS A LINEAR DETECTOR.

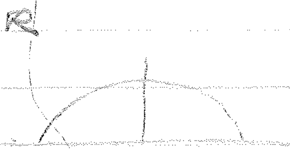
LAPLACE NOISE: $f(x) = \frac{\gamma}{2} e^{-\gamma|x|}$

$$\ln f(x) = \ln\left(\frac{\gamma}{2}\right) - \gamma|x|$$

$$-\frac{d}{dx} \ln f(x) = \gamma \,\mathrm{sgn}(x)$$

$$\therefore g_{lo}(x) = \gamma \,\mathrm{sgn}(x)$$

NOTE: THIS IS A SIGN DETECTOR

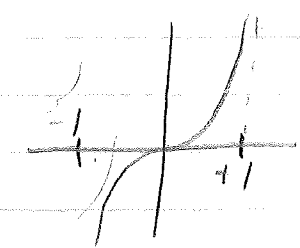


BETA NOISE: $f(x) = \frac{3}{4}(1-x^2)$

$$\ln f(x) = \ln 3/4 + \ln(1-x^2)$$

$$-\frac{d}{dx} \ln f(x) = \frac{2x}{1-x^2}$$

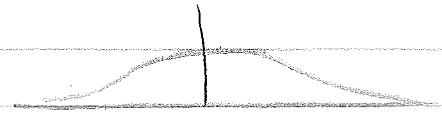
$$\therefore g_{lo}(x) = \frac{2x}{1-x^2} \Longrightarrow$$



10-8-75 (WED)

NOCALLY OPTIMUM

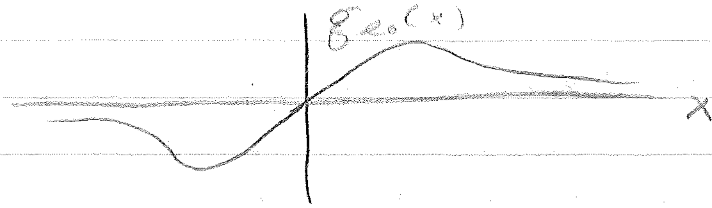
- CAUCHY NOISE



$$f(x) = \frac{m}{\pi(m^2+x^2)}$$

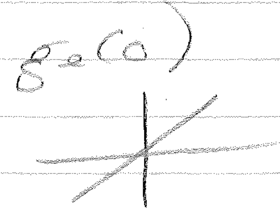
$$\ln f(x) = \ln \frac{m}{\pi} - \ln(m^2+x^2)$$

$$-\frac{d}{dx}\ln f(x) = \frac{2x}{m^2+x^2} = g_{eo}(x)$$

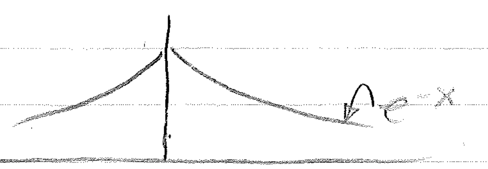


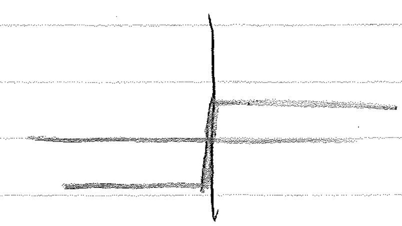
- GAUSSIAN NOISE: $g_{eo}(x) = x/\sigma^2$
- LAPLACE NOISE: $g_{eo}(x) = \gamma\,\mathrm{sgn}(x)$
- BETA NOISE: $g_{eo}(x) = \frac{2x}{1-x^2}$
- CAUCHY NOISE: $g_{eo} = \frac{2x}{m^2+x^2}$

- GAUSS.

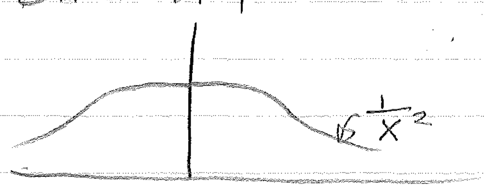


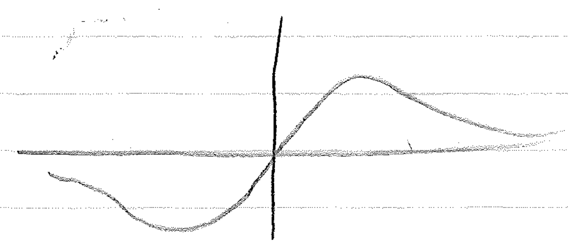
- LAPLACE



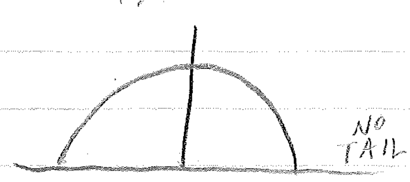


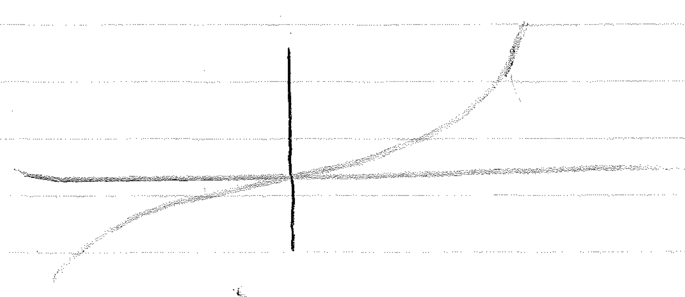
- CAUCHY





- BETA





EXAMPLE: LOCALLY OPTIMAL N.P. DETECTOR IN LAPLACE NOISE $\sim \frac{1}{2} e^{|n|}$. WE HAVE 1 OBSERVATION.

TEST STATISTIC (t) IS $\mathrm{sgn}(Y)$.

GIVEN A VALUE OF α $\alpha < 1/2$.

$$\alpha = E_0[\phi(x)]$$

$$\phi(x) = \begin{cases} 1 & ; \mathrm{sgn}(Y) > T \\ p & ; \mathrm{sgn}(Y) = T \\ 0 & ; \mathrm{sgn}(Y) < T \end{cases}$$

WE NEED TO RANDOMIZE.

PICK $T = 1$

$$\alpha = E_0[\phi(x)] = 1 \cdot 0 + p \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{p}{2} = \alpha$$

$$\Longrightarrow p = 2\alpha$$

DECISION RULE: IF OBSERVATION H_1 WITH PROBABILITY $P = 2\alpha$.

NOTE: FOR $\alpha = \frac{1}{2}$, PICK $T \in (-1, 1)$. LET $T = 0$. FOR $1 > \alpha > \frac{1}{2}$, LET $T = -1$.

EXAMPLE: SAME AS ABOVE, EXCEPT FOR TWO OBSERVATIONS.

TEST STATISTIC $t \triangleq \mathrm{sgn}(Y_1) + \mathrm{sgn}(Y_2)$

$$P_0(t = 2) = 1/4$$
$$P_0(t = 0) = 1/2$$
$$P(t = -2) = 1/4$$

SAY $\alpha < \frac{1}{4} \Rightarrow$ SET $T = 2$ & RANDOMIZE

$$\alpha = p\left(\tfrac{1}{4}\right) \Rightarrow p = 4\alpha$$

DECISION RULE: IF BOTH OBSERVATIONS ARE POSITIVE, ANNOUNCE H_1 WITH PROBABILITY $P = 4\alpha$

FOR $\alpha = \frac{1}{4} \Rightarrow$ SET $T \in (0,2)$ (NO RANDOMIZING) SO LET $T = 1$. DECISION RULE: IF BOTH OBSERVATIONS ARE POSITIVE, ANNOUNCE H_1

NOTE: β ONLY FOUND WITH KNOWLEDGE OF SIGNAL. BUT IF SIGNAL WERE KNOWN, WE'D USE NORMAL N.P.

EXAMPLE: LOCALLY OPTIMAL N-P DETECTOR IN STATIONARY WHITE GAUSSIAN NOISE $\sim N(0,\sigma^2)$

TEST STATISTIC IS $\sum_{i=1}^{k} Y_i$

TEST IS $S = \sum_{i=1}^{k} Y_i \gtrless T$.

UNDER NULL HYPOTHESIS,

$$S \sim N(0, k\sigma^2)$$

$$\alpha = \int_T^\infty \frac{1}{\sqrt{2\pi}\,k\sigma} e^{-\frac{x^2}{2k\sigma^2}}\,dx$$

LET $U = \frac{x}{\sqrt{k}\,\sigma}$

$$\Rightarrow \alpha = \int_{\frac{T}{\sqrt{k}\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$$

$$= 1 - \int_{-\infty}^{\frac{T}{\sqrt{k}\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du$$

$$= 1 - \Phi\left(T/\sqrt{k}\,\sigma\right)$$

$$T = \sqrt{k}\,\sigma\,\Phi^{-1}(1-\alpha)$$

SAME AS FOR REGULAR N-P TEST!

10-10-75 (FRI)

IN CLASS, CLOSED BOOK TEST (NO PROOFS)

EXAMPLE: CONSIDER USING A N.P. DETECTION SCHEME FOR A UNIT SIGNAL IN STATIONARY WHITE GAUSSIAN NOISE WITH ZERO MEAN AND UNIT VARIANCE. SET $\alpha = 0.01$. HOW MANY OBSERVATIONS MUST WE TAKE SO THAT $\beta \geq 0.99$?

FROM BEFORE, THE TEST IS

$$S = \sum_{i=1}^{K} Y_i \underset{H_0}{\overset{H_1}{\gtrless}} T$$

UNDER THE NULL HYPOTHESIS,

$$S \sim N(0, K)$$

$$\alpha = 0.01 = \int_T^\infty \frac{1}{\sqrt{2\pi K}} e^{-\frac{x^2}{2K}} dx$$

$$= \int_{T/\sqrt{K}}^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= 1 - \Phi(T/\sqrt{K})$$

$$\Phi(T/\sqrt{K}) = 0.99$$

$$T = \sqrt{K}\, \Phi^{-1}(0.99)$$

$$\simeq 2.33 \sqrt{K}$$

UNDER HYPOTHESIS H_1

$$S \sim N(\underset{\mu}{K}, \underset{\sigma^2}{K})$$

$$\beta = \int_T^{\infty} P_1(s)\,ds = 0.99$$

$$= \int_{2.33\sqrt{K}}^{\infty} \frac{1}{\sqrt{2\pi K}} e^{\frac{-(x-K)^2}{2K}} dx$$

LET $U = \frac{x-K}{\sqrt{K}}$

$$\Rightarrow \beta = \int_{2.33-\sqrt{K}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{U^2}{2}} dU$$

$$= 1 - \int_{-\infty}^{2.33-\sqrt{K}} \frac{1}{\sqrt{2K}} e^{-U^2/2} dU$$

$$= 1 - \Phi(2.33 - \sqrt{K}) \geq 0.99$$

$$\Phi(2.33 - \sqrt{K}) \leq 0.01$$

$$2.33 - \sqrt{K} \leq \Phi^{-1}(0.01) \simeq -2.33$$

$$\sqrt{K} \geq 4.66$$

$$K \geq 21.7 \geq 22$$

$$\therefore K = 22$$

"GENERALIZED" GAUSSIAN NOISE

$H_0: Y_i = n_i$

$H_1: Y_i = S_i + n_i$ $\qquad f(n) = k e^{-K_1 n^4}$

TAILS FALL FASTER THAN GAUSSIAN

$$g(Y) = \ln\left[\frac{f(Y-S)}{f(S)}\right]$$

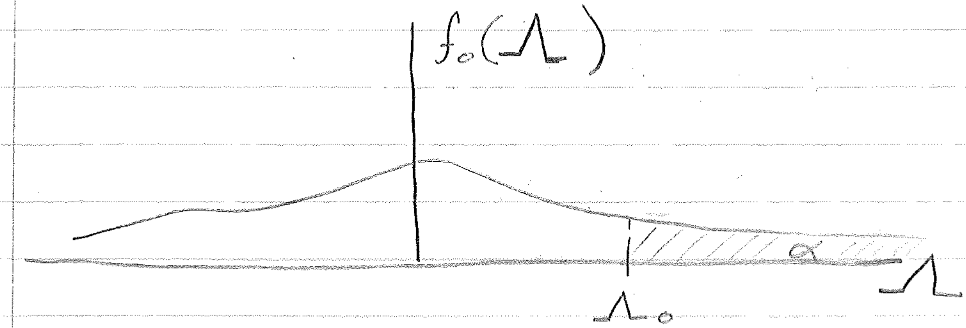
$$= -K_1\left[(Y-S)^4 - Y^4\right]$$

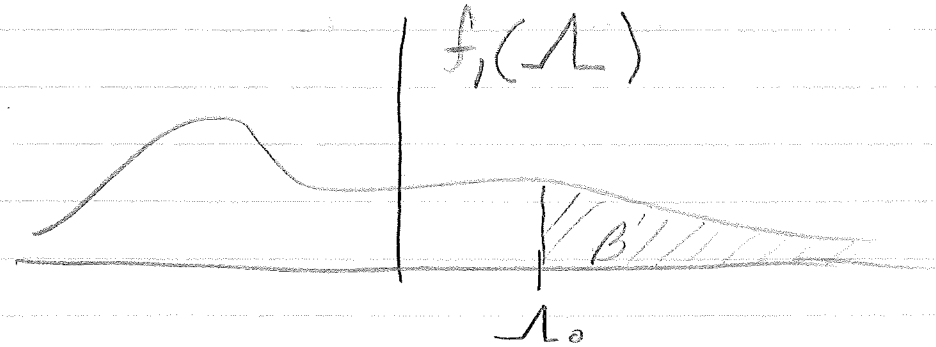
$$= -K_1\left[Y^4 - 4Y^3S + 6Y^2S^2 - 4YS^3 + S^4 - Y^4\right]$$

$$= -K_1\left[4Y^3S - 6Y^2S^2 + 4YS^3 + S^4\right]$$

ZMNL IS A CUBIC

NOTE: FOR A N.P. TEST, AS LONG AS DENSITY FUNCTION OF THE LIKLIHOOD RATIO EXISTS UNDER H_0 (ie, NO DIRAC DELTAS), NO NEED FOR RANDOMIZATION.





10-13-75 (MON)

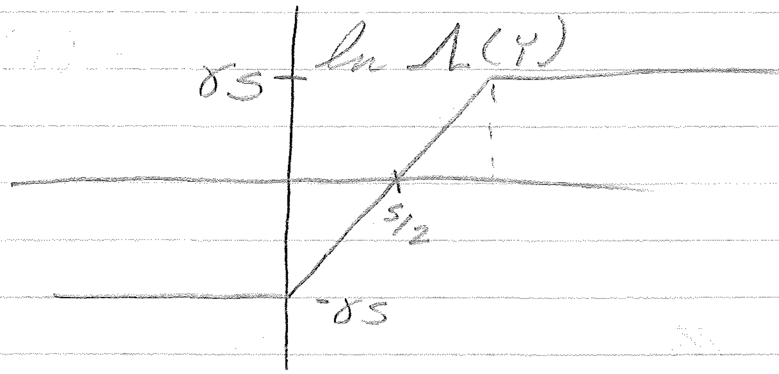
TEST #1

10-15-75 (WED)

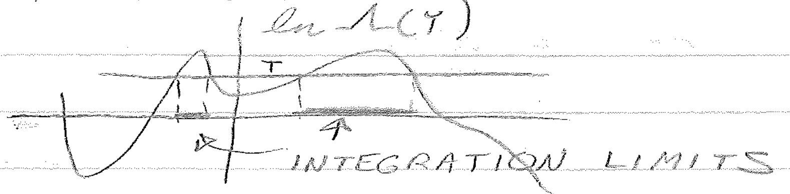
TEST ANSWERS

1a. $\alpha = \int_{S/2}^{\infty} \frac{\delta}{2} e^{-\delta x}\, dx = \frac{1}{2} e^{-\delta S/2}$

b.

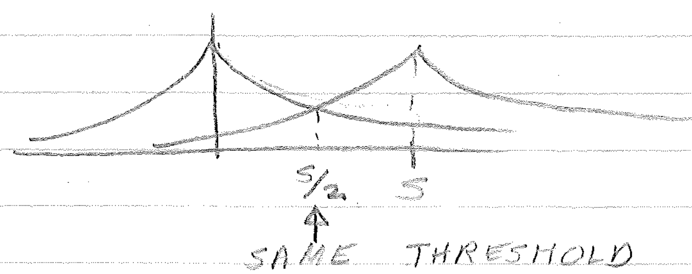


CONSIDER GENERAL CASE



TEST $Y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{S}{2}$

c. $P_1(Y)/P_0(Y) \gtrless \frac{\Pi_0}{\Pi_1} = 1$



2. $t_i = \text{sgn}[Y_i]$

$$t = \sum_{i=1}^{3} \text{sgn}\, Y_i \gtrless T$$

$$P_0[3 = \Lambda] = \frac{1}{8}$$
$$P_0[1] = 3/8$$
$$P_0[-1] = 3/8$$
$$P[-3] = \frac{1}{8}$$
$$T \in (1,3) \Rightarrow \alpha = \frac{1}{8}$$

3. $S = \sum_{i=1}^{K} Y_i$ UNDER H_0, $S \sim N(0, K\sigma^2)$

UNDER H_1, $S \sim N(KS, K\sigma^2)$

$$\alpha = \int_T^\infty \frac{1}{\sqrt{2\pi K}} e^{-x^2/2K\sigma^2} dx$$

$$= \int_{\frac{T}{\sqrt{K}\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2} d\mu$$

$$1 - \alpha = \int_{-\infty}^{T/\sqrt{K}\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \Phi\left(\frac{T}{\sqrt{K}\sigma}\right)$$

$$\Rightarrow T = \sigma\sqrt{K}\, \Phi^{-1}(1-\alpha)$$

$$\beta = \int_T^\infty \frac{1}{\sqrt{2\pi K}\sigma} e^{-\frac{(x-KS)^2}{2K\sigma^2}} dx$$

$$= \int_{\frac{T-KS}{\sqrt{K}\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{\Phi^{-1}(1-\alpha) - \sqrt{K}\, S/\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= 1 - \Phi\left[\Phi^{-1}(1-\alpha) - \frac{\sqrt{K}\, S}{\sigma}\right]$$

REVIEW OF RANDOM PROCESSES

A RANDOM PROCESS (OR STOCHASTIC PROCESS) IS AN INDEXED FAMILY OF RANDOM VARIABLE $\{X(t), t \in T\}$, t IS INDEX PARAMETER, T IS AN INDEX SET. WE WILL TAKE T TO BE THE REAL LINE OR AN INTERVAL AND ASSOCIATE THE PARAMETER t WITH TIME. $X(t) = X(t, \omega)$. $t \in T$, $\omega \in \Omega \ni \Omega$ IS THE PROBABILITY SPACE.

FIXING $t \Longrightarrow$ RANDOM VARIABLE.

FIXED $\omega \Longrightarrow$ FUNCTION OF TIME, A REALIZATION OR SAMPLE FUNCTION OF THE RANDOM PROCESS. $X(t)$ IS SAID TO BE A SECOND ORDER RANDOM PROCESS, IF

$$E\{[X(t)]^2\} < \infty.$$

WE WILL ASSUME THAT $X(t)$ IS A SECOND ORDER RANDOM PROCESS.

AUTOCORRELATION FUNCTION

$$R(t, s) = E\{X(t)\, X(s)\}$$

NOTE: $R(t, s) = R(s, t)$

$R(t, s)$ IS NON-NEGATIVE DEFINATE.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j R(t_i, t_j) \geq 0$$

WHERE a_i AND a_j ARE ANY CONSTANTS.

PROOF:

$$E\left[\left(\sum_{i=1}^{n} a_i\, x(t_i)\right)^2\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\, X(t_i) X(t_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\, R(t_i, t_j) \geq 0$$

FOR ANY ARBITRARY FINITE SUBSET OF T, SAY $t_1, t_2, \ldots, t_n$, THE RANDOM VARIABLES $X(t_1), X(t_2), \ldots, X(t_n)$ WILL HAVE A FINITE DIMENSIONAL DISTRIBUTION FUNCTION GIVEN BY

$$F(x_1, x_2, \ldots x_n; t_1, t_2, \ldots t_n)$$

$$= P\left[X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots X(t_n) \leq x_n\right]$$

THE COLLECTION OF ALL FINITE DIMENSIONAL DISTRIBUTION FUNCTIONS SERVES AS A COMPLETE STATISTICAL DESCRIPTION OF THE RANDOM PROCESS.

10-17-75 (FRI)

REVIEW OF L_2 THEORY

LET L_2 DENOTE THE SPACE OF ALL (MEASURABLE) FUNCTIONS THAT ARE SQUARE INTEGRABLE ON $[a, b]$:

$$\int_a^b [f(t)]^2 dt < \infty$$

WE WILL TAKE a AND b TO BE FINITE.

DEFINE THE INNER PRODUCT AS

$$(f, g) = \int_a^b f(t) g(t) dt$$

DEFINE THE NORM AS

$$\| f \| = \sqrt{(f, f)} = \sqrt{\int_a^b [f(t)]^2 dt}$$

WITH THIS NORM AND INNER PRODUCT, L_2 BECOMES A HILBERT SPACE. IF $\| f \| = 1$, THEN $f(t)$ IS SAID TO BE NORMALIZED. IF $(f, g) = 0$, THEN $f(t)$ AND $g(t)$ ARE SAID TO BE ORTHOGONAL.

A CLASS OF FUNCTIONS $\{f_n(t)\}$ BELONGING TO L_2 IS SAID TO BE AN ORTHOGONAL CLASS IF

$$(f_i, f_h) = 0 \quad \forall \; i \neq h$$

IT IS AN ORTHONORMAL CLASS,

IN ADDITION, $\|f_i\| = 1$

ONE MAY ALWAYS NORMALIZE: $\tilde{f}_n(t) = \frac{f_n(t)}{\|f_n\|}$

UNLESS OTHERWISE STATED, CONVERGENCE IS TAKEN TO TO BE IN THE L_2 NORM,

ie $g(t) = \sum_{n=1}^{\infty} a_n f_n(t)$ MEANS

$$\lim_{N\to\infty} \int_a^b \left[g(t) - \sum_{n=1}^{N} a_n f_n(t)\right]^2 dt = 0$$

OR EQUIVALENTLY

$$\lim_{N\to\infty} \left\|g - \sum_{n=1}^{N} a_n f_n\right\| = 0 .$$

NOTE: IF $\{f_n(t)\}$ ARE ORTHONORMAL, THEN $a_n = (g, f_n) = \int_a^b g(t) f_n(t) dt$

CONSIDER A FUNCTION $h(t) \in L_2$. LET $h_n = (h, f_n)$ WHERE $\{f_n(t)\}$ IS AN ORTHONORMAL CLASS OF FUNCTIONS.

$$\Longrightarrow \|h\|^2 \geq \sum_{n=1}^{\infty} (h_n)^2 \Longleftarrow \text{BESSEL'S INEQUALITY}$$

$$\text{ie } \int_a^b |h(t)|^2 dt \geq \sum_{n=1}^{\infty} \left(\int_a^b h(t) f_n(t) dt\right)^2$$

(PARCEVAL'S THEOREM IS SPECIAL CASE)

THE CLASS OF ORTHONORMAL FUNCTIONS $\{f_n(t)\}$ IS SAID TO BE COMPLETE IF ANY FUNCTION $h(t) \in L_2$ CAN BE EXPRESSED AS

$$h(t) = \sum_{n=1}^{\infty} a_n f_n(t).$$

THIS IS EQUIVALENT TO SAYING THAT $\forall$ FUNCTION IN L_2 BESSEL'S INEQUALITY IS AN EQUALITY. THAT IS

$$\|h\|^2 = \sum_{n=1}^{\infty} (h_n)^2$$

THIS IS A SPECIAL CASE OF PARCEVAL'S THEOREM.

PARCEVAL'S THEOREM: IF $\{f_n(t)\}$ IS A COMPLETE ORTHONORMAL SET AND IF $g(t) \in L_2$ AND $h(t) \in L_2$, THEN

$$(g, h) = \sum_{n=1}^{\infty} g_n h_n, \text{ WHERE}$$

$$g_n = (g, f_n) \text{ AND } h_n = (h, f_n)$$

$$\text{i.e.: } \int_a^b g(t) f(t)\,dt = \sum_{n=1}^{\infty} g_n h_n$$

10-20-75

MERCER'S THEOREM

ANY FUNCTION, $R(t,s)$, WHICH IS SYMMMETRIC, CONTINUOUS, AND NON-NEGATIVE DEFINITE, IN THE SQUARE $a \le t \le b$, $a \le s \le b$, MAY BE EXPANDED IN THE ABSOLUTELY AND UNIFORMLY CONVERGENT SERIES

$$R(t,s) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t)\phi_n(s) : \begin{array}{l} a \le t \le b \\ a \le s \le b \end{array}$$

HERE THE SET OF FUNCTIONS $\phi_n(t)$ AND THE SET OF CONSTANTS, $\{\lambda_n\}$ ARE CALLED THE EIGENFUNCTIONS AND EIGENFUNCTIONS, RESPECTIVELY OF THE HOMOGENEOUS INTEGRAL EQN:

$$\lambda \phi(s) = \int_a^b R(t,s)\phi(t)\,dt$$

THE $\phi_n(t)$ ARE TAKEN TO BE ORTHONORMAL IN $[a,b]$

FOR PROOF, SEE a. COURANT & HILBERT "METHODS OF MATH. PHYS, VOL 1, p. 138 OR RIESY-SY NAGY, FUNCTIONAL ANALYSIS, p. 245.

NOTE: a. $f(t)$ IS NON-NEGATIVE DEFINATE IF $\mathfrak{F}[f(t)] \ge 0 \;\forall\; f$.

b. A MATRIX IS NON-NEGATIVE DEFINATE IF IT CAN BE DIAGONALIZED.

KARHUNEN - LOEVE EXPANSION:

LET $x(t)$ BE A (SECOND ORDER) RANDOM PROCESS HAVING A CONTINUOUS AUTOCORRELATION FUNCTION. IT MAY BE REPRESENTED BY:

$$X(t) = \sum_{n=1}^{\infty} x_n \phi_n(t) \quad , \quad a \le t \le b$$

WHERE $x_n = \int_a^b X(t)\, \phi_n(t)\, dt$, AND WHERE THE CONVERGENCE IS IN THE MEAN SQUARE SENSE. ALSO, THE COEFFICIENTS x_n, ARE UNCORRELATED.

MEAN SQUARE CONVERGENCE:

$$\lim_{N \to \infty} E\left\{ \left[X(t) - \sum_{n=1}^{N} x_n \phi_n(t) \right]^2 \right\} = 0$$

PROOF: PROVE x_n ARE UNCORRELATED;

$$E[x_n, x_m] = E\left[\int_a^b X(t)\phi_n(t)\,dt \int_a^b X(s)\,\phi_m(s)\,ds\right]$$

$$= \int_a^b \int_a^b E[X(t)X(s)]\,\phi_m(t)\phi_m(s)\,dt\,ds$$

$$= \int_a^b \underbrace{\int_a^b R(t,s)\,\phi_n(t)\,dt}\,\phi_m(s)\,ds$$

$$= \int_a^b \lambda_n \phi_n(s)\phi_m(s)\,ds$$

$$= \lambda_n \int_a^b \phi_n(s)\phi_m(s)\,ds = \lambda_n \delta_{nm}$$

$$E[x_n, x_m] = \begin{cases} 0 & ; \quad n \ne m \\ \lambda_n & ; \quad n = m \end{cases}$$

NOTE $\lambda_n = E[x_n^2] \geq 0$

10-22-75 (WED)

CONVERGENCE IN MEAN SQUARE:

LET

$$\bar{X}_n(t) = \sum_{n=1}^{N} x_n \phi_n(t)$$

BY CONVERGENCE IN MEAN SQUARE, WE MEAN

$$\lim_{N\to\infty} E[(\bar{X}(t) - \bar{X}_N(t))^2] = 0$$

NOW

$$E[(\bar{X}(t) - \bar{X}_N(t))^2] = E[|X(t)|^2] + E[|\bar{X}_N(t)|]^2 - 2E[X(t)\bar{X}_N(t)]$$

$$\equiv J_N$$

$$J_N = R(t,t) + \sum_{n=1}^{N}\sum_{m=1}^{N} E[X_n X_m]\phi_n(t)\phi_m(t) - 2\sum_{n=1}^{N} E[\bar{X}(t) x_n]\phi_n(t)$$

$$= R(t,t) + \sum_{n=1}^{N} \lambda_n \phi_n(t)\phi_n(t)$$

$$- 2\sum_{n=1}^{N} \int_a^b E[\bar{X}(t)\bar{X}(s)]\phi_n(s)ds \cdot \phi_n(t)$$

$$= R(t,t) + \sum_{n=1}^{N} \lambda_n \phi_n(t)\phi_n(t)$$

$$- 2\sum_{n=1}^{N} \int_a^b R(t,s)\phi_n(s)ds\,\phi_n(t)$$

$$= R(t,t) + \sum_{n=1}^{N} \lambda_n \phi_n(t)\phi_n(t) - 2\sum_{n=1}^{N} \lambda_n \phi_n(t)\phi_n(t)$$

$$= R(t,t) + \sum_{n=1}^{N} \lambda_n[\phi_n(t)]^2 - 2\sum_{n=1}^{N} \lambda_n[\phi_n(t)]^2$$

$$= R(t,t) - \underbrace{\sum_{n=1}^{N} \lambda_n[\phi_n(t)]^2}$$

CONVERGES POINTWISE TO $R(t,t)$ BY MERCER'S THEM.

$$\Rightarrow \lim_{N\to\infty} J_N = 0$$

AND $\bar{X}_n(t)$ CONVERGES IN MEAN SQUARE

CONSIDER THE INTEGRAL EQUATION

$$\lambda_n \phi_n(t) = \int_a^b R(t;s)\, \phi_n(s)\, ds$$

RECALL $\phi_n(t)$ ARE ORTHOGONAL.
IT TURNS OUT THAT:
$\sum_{n=1}^{\infty} (\lambda_n)^2 < \infty$ ← TRY TO SHOW.

WE WILL ASSUME THAT $R(t,s)$ IS POSITIVE DEFINATE (OPPOSED TO NON-NEGATIVE DEFINATE). THIS IS TRUE FOR "HONORABLE" NOISE. (SEC 4.4 IN HELSTRUM)
$\overset{\text{IFF}}{\Longleftrightarrow}$ $[\phi_n(t)]$ ARE COMPLETE IN L_2 ← TRY TO SHOW.

<u>SOLUTION TO AN INTEGRAL EQN</u>

LET $R(t_1, t_2)$ BE AN AUTOCORRELATION FUNCTION. CONSIDER THE INTEGRAL EQN. AS FOLLOWS:

$s(t_1) = \int_a^b R(t_1, t_2)\, q(t_2)\, dt_2$

R AND S ARE KNOWN, q IS NOT KNOWN. (NOTE $\int_a^b R(t_1,t_2)\, q(t_2)\, dt_2$ ~ MATCHED FILTER.)
THE EIGENFUNCTIONS AND EIGENVALUES OR $R(t_1,t_2)$ ARE GIVEN BY

$$\lambda_n \phi_n(t_1) = \int_a^b R(t_1, t_2)\, \phi_n(t_2)\, dt_2$$

ASSUME R IS POSITIVE DEFINATE
$\Longrightarrow \{\phi_n(t)\}$ IS COMPLETE IN L_2

ASSUME THAT $s(t)$ AND $q(t)$ BELONG TO L_2 (i.e. HAS FINITE ENERGY)

$$\left.\begin{aligned} s(t) &= \sum_{n=1}^{\infty} s_n \phi_n(t) \\ q(t) &= \sum_{n=1}^{\infty} q_n \phi_n(t) \end{aligned}\right\} \begin{array}{l} L_2 \text{ CONVERGENCE} \\ \text{(NOT POINTWISE)} \end{array}$$

$$s_n = (s, \phi_n) = \int_a^b s(t)\phi_n(t)dt$$
$$q_n = (q, \phi_n) = \int_a^b q(t)\phi_n(t)dt$$

SUBSTITUTE EXPANSIONS INTO INTEGRAL EQN:

$$\sum_{n=1}^{\infty} s_n \phi_n(t) = \int_a^b R(t_1, t_2)\left[\sum_{n=1}^{\infty} q_n \phi_n(t_2)\right]dt_2$$

(THIS IS AN L_2 CONVERGENCE)

(TRY TO SHOW: $\int_a^b (\quad)$ CONVERGES POINTWISE)

CAN SHOW UNIFORM CONVERGENCE OF RIGHT HAND SIDE. OK TO INTERCHANGE $\int$ AND Σ (TRY TO SHOW, USE SCHWARTZ'S INEQUALITY)

$$\sum_{n=1}^{\infty} s_n \phi_n(t) = \sum_{n=1}^{\infty} q_n \int_a^b R(t_1, t_2)\phi_n(t_2)dt_2$$
$$= \sum_{n=1}^{\infty} q_n \lambda_n \phi_n(t_1)$$

SINCE $\{\phi_n(t)\}$ IS AN ORTHONORMAL SET:

$$s_n = q_n \lambda_n \Rightarrow q_n = s_n / \lambda_n$$

s_n, q_n, AND λ_n ARE SQUARE SUMMABLE.

$$q(t) = \sum_{n=1}^{\infty} q_n \phi_n(t) = \sum_{n=1}^{\infty} \left(\frac{s_n}{\lambda_n}\right)\phi_n(t)$$

SUMMARY:

DEFINE $S_N(t) = \sum_{n=1}^{N} S_n \phi_n(t)$ AND

$q_N(t) = \sum_{n=1}^{N} q_n \phi_n(t) = \sum_{n=1}^{N} \frac{S_n}{\lambda_n} \phi_n(t)$.

THEN

$$S_N(t_1) = \int_a^b R(t_1, t_2)\, q_N(t_2)\, dt_2$$

A NECESSARY AND SUFFICIENT CONDITION THAT $q_N(t) \to q(t) \in L_2$ IS THAT $\sum_{n=1}^{\infty} (S_n/\lambda_n)^2 < \infty$

THIS ALSO GUARANTEES THAT $s(t) \in L_2$ AND

$$s(t_1) = \int_a^b R(t_1, t_2)\, q(t_2)\, dt_2$$

HAS A UNIQUE L_2 SOLUTION.

(NOTE CONVERSE IS NOT NECESSARILY TRUE)

10-24-75 (FRI)

DERIVATION OF THE TEST STATISTIC

$H_0: Y(t) = n(t)$ $\qquad a \le t \le b$

$H_1: Y(t) = s(t) + n(t)$

ASSUME 0 MEAN GAUSSIAN NOISE WITH A CONTINUOUS POSITIVE DEFINATE AUTO-CORRELATION FUNCTION, $R(t,s)$.

DEFINE

$$S_K = \int_a^b s(t)\phi_K(t)\,dt = (s, \phi_K)$$
$$n_K = \int_a^b n(t)\phi_K(t)\,dt = (n, \phi_K)$$
$$Y_K = \int_a^b Y(t)\phi_K(t)\,dt = (Y, \phi_K)$$

(CONT)

NOTE: $E_0[Y_K] = E[n_K] = 0$ (ie ZERO MEAN

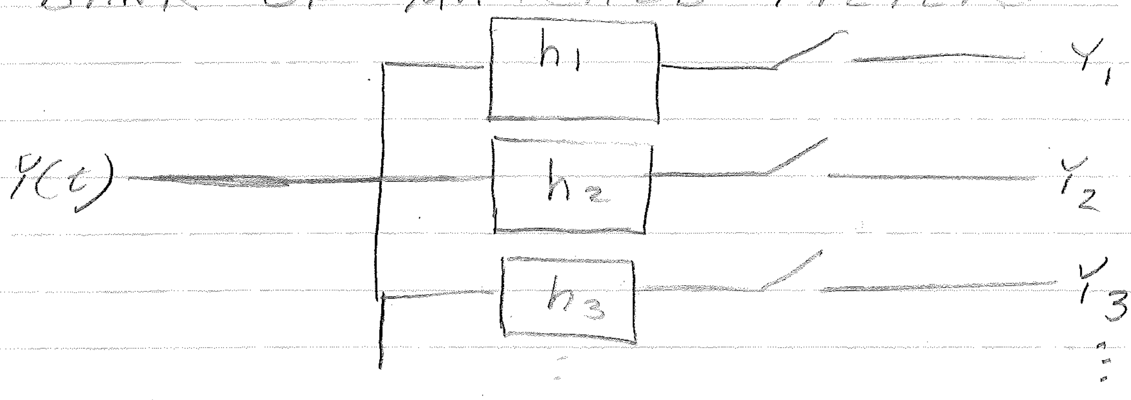
$E_1[Y_K] = E[S_K + n_K] = S_K$

$$cov(X,Y) \triangleq E\{[X-E(X)][Y-E(Y)]\}$$

$$cov(Y_K, Y_j) = \lambda_K \delta_{ij} \text{ UNDER EITHER HYPOTHESIS}$$
$$= cov(n_K, n_j)$$

WE WILL TAKE AS OUR OBSERVABLES
$\{Y_K\}, K = 1, 2, \ldots, \bar{K}$
UNDER EITHER HYPOTHESIS, WE HAVE
GAUSSIAN RANDOM VARIABLES

NOTE: CAN GET $\{Y_K\}, K = 1, 2, \ldots \bar{K}$
BY PUTTING $Y(t)$ THROUGH A
BANK OF MATCHED FILTERS



(c.)

(CONT)

$$H_0: P_0(Y_1, Y_2, \ldots, Y_K) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi \lambda_k}} \exp \frac{-Y_k^2}{2\lambda_k}$$

$$H_1: P_1(Y_1, Y_2, \ldots, Y_K) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp \frac{-(Y_k - S_k)^2}{2\lambda_k}$$

$$\Lambda(Y_1, Y_2, \ldots Y_K) = \frac{P_1(Y_1, Y_2, \ldots, Y_K)}{P_0(Y_1, Y_2, \ldots, Y_K)}$$

$$= \exp\left[\sum_{k=1}^{K} \frac{1}{2\lambda_k}\left[Y_k^2 - (Y_k - S_k)^2\right]\right]$$

$$= \exp \sum_{k=1}^{K} \frac{1}{2\lambda_k}\left[2Y_k S_k - S_k^2\right]$$

$$\Lambda \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$

LET $G_K \triangleq \sum_{k=1}^{K} S_k Y_k / \lambda_k$

TEST IS:

$$G_K = \sum_{1}^{K} \frac{S_k Y_k}{\lambda_k} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0 + \sum_{k=1}^{K} \frac{S_k^2}{2\lambda_k}$$

THIS IS TEST FOR K OBSERVABLES.
WHAT HAPPENS AS $K \to \infty$.

WE HAVE $E_0[G_K] = 0$

$E_1[G_K] = \sum_{k=1}^{K} S_k^2 / \lambda_k$

ALSO $\operatorname{Var}(G_K) = \sum_{k=1}^{K} S_k^2 / \lambda_k$

(UNDER BOTH HYPOTHESIS)

WE KNOW THAT G_K IS GAUSSIAN AND IS THE SUM OF INDEPENDENT RANDOM VARIABLES.

SUPPOSE $\sum_{k=1}^{\infty} \frac{S_k^2}{\lambda_k} < \infty$

ie: $\lim_{K \to \infty} \operatorname{Var}(G_K) < \infty$

$\Rightarrow$ THERE EXISTS A RANDOM VARIABLE $G \ni G_K \to G$ IN THE MEAN SQUARE SENSE.

SINCE WE HAVE INDEPENDENCE AND MEAN SQUARE CONVERGENCE, IT CAN BE SHOWN THAT THE CONVERGENCE IS ALSO WITH PROBABILITY ONE. (FOR PROOF, SEE DOOB, STOCHASTIC PROCESSES, pg 108)

10-27-75 (MON)

$$G_{\bar{K}} = \sum_{K=1}^{\bar{K}} S_K Y_K / \lambda_K$$

DEFINE $q_K = S_K / \lambda_K$

$$\Rightarrow G_{\bar{K}} = \sum_{K=1}^{\bar{K}} q_K Y_K$$

WE KNOW THAT $G_K \to G$ IN MEAN SQUARE & WITH PROBABILITY ONE

$$G_{\bar{K}} = \sum_{K=1}^{\bar{K}} q_K Y_K \Longrightarrow \sum_{K=1}^{\infty} q_K Y_K \quad \text{WP1 AND IN MS.}$$

NOTE: $\displaystyle\sum_{n=1}^{\infty} q_K Y_K = \int_a^b q(t)\, Y(t)\, dt$

i.e: $\displaystyle Y(t) = \sum_{K=1}^{\infty} Y_K \phi_K(t)$

$$q(t) = \sum_{K=1}^{\infty} q_K \phi_K(t)$$

ASSUME $\displaystyle\sum_{K=1}^{\infty} q_K^2 < \infty$

RECALL: TERM ON THE OTHER SIDE OF THE TEST WAS

$$\sum_{K=1}^{\infty} S_K^2/\lambda_K$$

(ie, FROM $G_K \gtrless \ln \Lambda_0 + \sum_{K=1}^{K} S_K^2/2\lambda_K$)

NOW: $\sum_{K=1}^{\infty} \frac{S_K^2}{\lambda_K} = \sum_{K=1}^{\infty} q_K S_K = \int_a^b q(t)s(t)\,dt$ ← FROM PARCEVAL

NOTE: $\sum_{n=1}^{\infty} \left(\frac{S_K}{\lambda_K}\right)^2 < \infty \Longrightarrow \sum_{K=1}^{\infty} \frac{S_K^2}{\lambda_K} < \infty$ (PROVE)

SUMMARIZE:

$$G[Y(t)] = \int_a^b q(t)Y(t)\,dt \quad \leftarrow \text{R.V. TRANSFORMATION}$$

$$G[Y(t)] \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0 + \frac{1}{2}\int_a^b s(t)q(t)\,dt$$

WHERE $s(t_1) = \int_a^b R(t_1, t_2)\, q(t_2)\, dt_2$

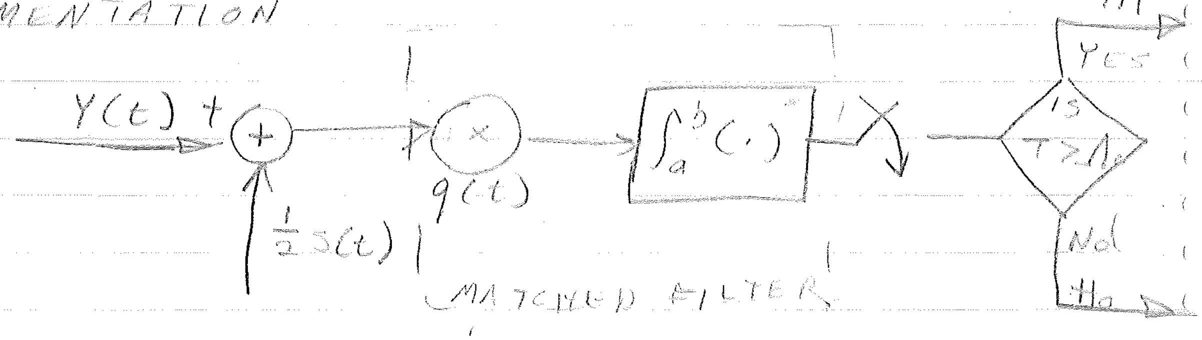
$q(t)$ IS A FLIPED AROUND AS A LINEAR TIME INVARIANT IMPULSE RES

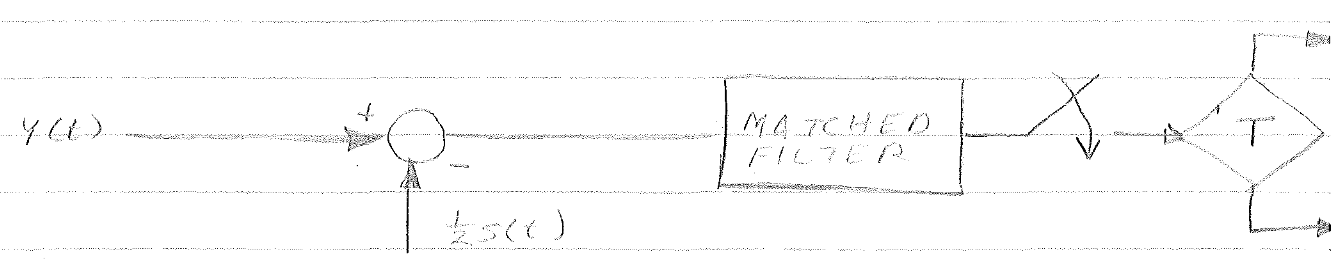
$q(t)$ IS MANY TIMES TERMED: "PSUEDO SIGNAL"

EQUIVALENTLY:

$$\int_a^b q(t)\left[Y(t) - \tfrac{1}{2}s(t)\right]dt \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0$$

IMPLIMENTATION





CONSIDER THE SIGNAL TO NOISE (POWER) RATIO. (SNR)

IF $Y(t) = S(t)$, THEN FILTER OUTPUT IS $\left[\int_a^b s(t)q(t)\,dt\right]^2$

IF $Y(t) = n(t)$, THEN NOISE POWER IS

$$E = \left\{\left[\int_a^b n(t)q(t)\,dt\right]^2\right\}$$

$$SNR = \frac{\left[\int_a^b s(t)q(t)\,dt\right]^2}{E\left[\int_a^b\int_a^b n(t_1)\,n(t_2)\,q(t)\,q(t_2)\,dt_1\,dt_2\right]}$$

$$= \frac{\left[\int_a^b s(t)q(t)\,dt\right]^2}{\int_a^b\int_a^b E\left[n(t_1)\,n(t_2)\right]q(t_1)\,q(t_2)\,dt_1\,dt_2}$$

$$= \frac{\left[\int_a^b s(t)q(t)\,dt\right]^2}{\int_a^b\int_a^b R(t_1,t_2)\,q(t_1)\,q(t_2)\,dt_1\,dt_2}$$

$$= \frac{\left[\int_a^b s(t)q(t)\,dt\right]^2}{\int_a^b s(t_2)\,q(t_2)\,dt_2}$$

$$= \int_a^b s(t)\,q(t)\,dt$$

$$= d^2 = \sum_{K=1}^{\infty} q_K S_K$$

$$= \sum_{K=1}^{\infty} \frac{S_K^2}{\lambda_K} = SNR$$

10-29-75 (WED) (NOT FOR TEST)

THE DETECTION PROBLEM FROM AN ABSTRACT VIEWPOINT

LET $\Omega(a,b)$ BE THE SET OF ALL REAL-VALUED FUNCTIONS DEFINED ON $[a,b]$.

LET $F[a,b]$ BE A MINIMAL σ-ALGEBRA OF SUBSETS $\Omega(a,b)$ THAT CONTAINS ALL CYLINDER SET.

THEN $[\Omega(a,b), F(a,b)]$ IS A MEASURABLE SPACE.

A RANDOM PROCESS, $X(t)$ DEFINED ON $F[a,b]$ A PROBABILITY MEASURE, μ, SUCH THAT

$$\mu(A) = P\{X(t,\omega) \in A\}$$

$\forall$ CYLINDER SET $A \in F(a,b)$

NOW, CONSIDER THE DETECTION PROBLEM:

LET μ_0 BE THE MEASURE INDUCED BY $Y(t)$ UNDER H_0, AND LET μ_1 BE THE MEASURE INDUCED BY $Y(t)$ UNDER H_0.

THE LIKELIHOOD RATIO IS THE FUNCTIONAL:

$$L(y(t)) = \frac{d\mu_1}{d\mu_0}[Y(t)]$$

IN THE PROBLEM THAT WE ARE CONSIDERING, WE HAVE SHOW THAT THIS EQUALS

$$\exp\left[\int_a^b q(t)\left[Y(t) - \tfrac{1}{2}s(t)\right]dt\right.$$
$$= \exp \sum_{k=1}^{\infty}\left[\frac{S_k Y_k}{\lambda_k} - \frac{S_k^2}{2\lambda_k}\right]$$

REFERENCE: SOROKHOD: STUDIES IN THE THEORY OF RANDOM PROCESSES, pg 97

DISTRIBUTION OF THE TESTSTATISTIC

$$G = \int_a^b q(t)\, Y(t)\, dt$$

UNDER H_0, $E_0(G) = 0$

$$Var_0(G) = \int_a^b\int q(t_1)q(t_2)R(t_1,t_2)\,dt_1 dt_2$$
$$= \int_a^b s(t)\,q(t) = d^2$$

(SIG/NOISE RATIO)

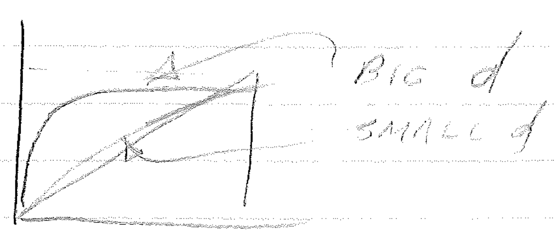
UNDER H_1: $E_1(G) = \int_a^b s(t)q(t)\,dt = d^2$

$$Var_1(G) = d^2$$

ie H_0: $G \sim N(0, d^2)$

H_1: $G \sim N(d^2, d^2)$

NEYMANN-PEARSON CURVE



10-31-75 (FRI)

CONSIDER TWO SIGNALS:

$$H_0 : Y(t) = s_0(t) + n(t)$$
$$H_1 : Y(t) = s_1(t) + n(t)$$

A SIMILAR DERIVATION AS BEFORE YIELDS

$$G = \int_a^b Y(t)\left[q_1(t) - q_0(t)\right]dt \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0 + \frac{1}{2}\int_a^b \left[s_1(t)q_1(t) - s_0(t)q_0(t)\right]dt$$

WHERE

$$s_0(t_1) = \int_a^b R(t_1, t_2)\, q_0(t_2)\, dt_2$$
$$s_1(t_1) = \int_a^b R(t_1, t_2)\, q_1(t_2)\, dt_2$$

NON-ZERO MEAN NOISE

CONSIDER CASE WHERE NOISE IS NOT ZERO MEAN. THEN WRITE IT AS $m(t) + n(t)$, WHERE $m(t)$ IS THE MEAN AND $n(t)$ IS ZERO MEAN NOISE, THEN ASSOCIATE $m(t)$ WITH THE SIGNAL.

ZERO MEAN STATIONARY WHITE GAUSSIAN NOISE:

$$R(t_1, t_2) = N\,\delta(t_1 - t_2)$$

$$s(t) = \int_a^b R(t_1, t_2)\, q(t_2)\, dt_2$$
$$= N \int_a^b \delta(t_1 - t_2)\, q(t_2)\, dt_2$$
$$= N\, q(t_1)$$
$$\therefore q(t) = \frac{1}{N} s(t)$$
$$\Rightarrow G = \frac{1}{N} \int_a^b Y(t)\, s(t)\, dt \underset{H_0}{\overset{H_1}{\gtrless}} T$$

EXAMPLE OF A KARHUNEN-LOEVE INTEGRAL EQUATION. BASIC INTERVAL $[0, T]$. DEFINE $R(t,s) = \text{MIN}\,(t,s)$
(BROWNIAN MOTION)

$$\lambda \phi(t) = \int_0^T R(t,s)\, \phi(s)\, ds$$
$$= \int_0^T \min(t,s)\, \phi(s)\, ds \quad (*)$$

$$= \int_0^t s\, \phi(s)\, ds + \int_t^T t\, \phi(s)\, ds$$

RECALL:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} F(x,t)\, dx$$

$$= \int_{a(t)}^{b(t)} \frac{\delta}{\delta t} F(x,t)\, dX$$
$$+ F[b(t), t] \cdot \frac{\delta b(t)}{\delta t}$$
$$- F[a(t), t] \cdot \frac{\delta a(t)}{\delta t}$$

GIVES

$$\lambda \phi'(t) = t\phi(t) + \int_t^T \phi(s)\,ds - t\phi(t)$$

$$= \int_t^T \phi(s)\,ds \qquad (**)$$

$$\lambda \ddot{\phi} = -\phi(t)$$

$$\ddot{\phi}(t) + \frac{1}{\lambda}\phi(t) = 0$$

GIVES:

$$\phi(t) = A \sin \frac{t}{\sqrt{\lambda}} + B \cos \frac{t}{\sqrt{\lambda}}$$

FROM INSPECTION OF $*$, WE SEE $\phi(0) = 0 \Rightarrow B = 0.$

$\therefore \phi(t) = A \sin t/\sqrt{\lambda}$

FROM $**$

$$\phi'(t)\big|_{t=T} = 0$$

$$\therefore \frac{A}{\sqrt{\lambda}} \cos \frac{T}{\sqrt{\lambda}} = 0$$

$$\Rightarrow \cos \frac{T}{\sqrt{\lambda}} = 0$$

$$\Rightarrow \frac{T}{\sqrt{\lambda_n}} = (n - \tfrac{1}{2})\pi \ ; \ n = 0, 1, 2$$

$$\Rightarrow \lambda_n = \frac{T^2}{(n-\frac{1}{2})^2 \pi^2}$$

REQUIRE $\int_0^T |\phi_1(t)|^2\,dt = 1$

$$\Rightarrow A^2 \int_0^T \sin^2\left(\frac{\pi t}{2T}\right) dt = A^2 \int_0^T \left[\frac{1}{2} - \frac{1}{2}\cos\frac{\pi t}{T}\right]$$

$$= A^2\, T/2 = 1 \Rightarrow A = \sqrt{2/T}$$

$$\therefore \phi_n(t) = \sqrt{2/T}\, \sin t/\sqrt{\lambda_n}$$

SUMMARIZE:

$$R(t,s) = \sum \lambda_n \phi_n(t)\phi_n(s) =$$

$$\min(t,s) = \sum_{n=1}^{\infty} \frac{2T}{\pi^2} \frac{\sin(t/\sqrt{\lambda_n})\sin(s/\sqrt{\lambda_n})}{(n-\frac{1}{2})^2}$$

CONVERGES UNIFORMLY (PT. WISE)

11-3-75 (MON)

TESTING COMPOSITE HYPOTHESES

e.g. H_0: $p_0(x)$

H_1: $p_1(x, \theta)$; θ = A PARAMETER, UNKNOWN

$P_1(x)$ IS COMPOSITE, $P_0(x)$ IS SIMPLE

CONSIDER BAYES TEST

ASSUME WE KNOW THE PRIOR PROBABILITIES (π_0 AND π_1), THE DENSITY OF θ, $z(\theta)$, AND THE COSTS C_{10}, C_{00}, $C_{01}(\theta)$, $C_{11}(\theta)$.

(C_{ij} = COST OF SAYING i WHEN j IS TRUE)

ASSUME $\begin{cases} C_{10} > C_{00} \\ C_{01}(\theta) > C_{11}(\theta) \end{cases}$ (i.e. CORRECT DECISION COST LESS

EXPECTED COST:

- COST OF $H_0/H_0 = \pi_0 \left[C_{00} \int_{R_0} p_0(x)dx \right]$
- " " $H_1/H_0 = \pi_0 \left[C_{10} \int_{R_1} p_0(x)dx \right]$
- " " $H_0/H_1 = \pi_1 \left[\int_{R_0} \int C_{01}(\theta) p_1(x,\theta) z(\theta) d\theta \, dx \right.$
- " " $H_1/H_1 = \pi_1 \left[\int_{R_1} \int C_{11}(\theta) p_1(x,\theta) z(\theta) d\theta \, dx \right.$

SUM OF THESE IS EXPECTED COST.

WE WANNA MINIMIZE EXPECTED COST. GOING THROUGH THE APPROPRIATE MATHEMATICS, WE GET THE BAYES TEST:

$$\frac{\pi_1 \int [C_{01}(\theta) - C_{11}(\theta)] z(\theta) \, P_1(x,\theta) d\theta}{\pi_0 \; [C_{11} - C_{00}] \, p_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

SPECIAL CASE: IF THE COSTS ARE INDEPENDENT OF PARAMETER VALUES, THEN

$$\Lambda(x;\theta) = \frac{P(x_1,\theta)}{P_0(x)}$$

$$\bar{\Lambda}(x) = \int \Lambda(x,\theta)\, z(\theta)\, d\theta \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})}$$

NOTE: CHOOSE $C_{01}=C_{10}=1$, $C_{00}=C_{11}=0$. AND THE BAYES TEST MINIMIZES THE PROBABILITY OF ERROR

RECALL: THE NEYMANN-PEARSON LEMMA WAS CONCERNED WITH SIMPLE HYPHOTHESIS. ASSUME WE KNOW $P_0(x)$ AND $P_1(x,\theta)$. FOR A FIXED VALUE OF θ, WE CAN FIND A N-P. TEST OF SIZE THAT MAXIMIZES TEST PWR.

$$\text{ie}\quad \frac{P_1(x,\theta)}{P_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

$$R_1(\theta) = \left\{ x;\ \frac{P_1(x;\theta)}{P_0(\theta)} > T \right\}$$

$$\alpha = \int_{R_1(\theta)} p_0(x)\,dx = \alpha(\theta)$$

IF R_1 IS INDEPENDENT OF θ, THE TEST IS CALLED A UNIFORMLY MOST POWERFUL TEST (UMP). THAT IS, THE SIZE IS α AND THE POWER IS MAXIMIZED, REGARDLESS OF θ.

EXAMPLE:

$H_0: \; x = n$

$H_1: \; x = \theta + n$ ASSUME $\theta > 0$

NOISE $\sim N(0, \sigma^2)$

$p_0(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$

$p_1(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\theta)^2/2\sigma^2}$

$$\frac{p_1(x;\theta)}{p_0(x)} = \exp\left(\frac{1}{2\sigma^2}\left[2x\theta - \theta^2\right]\right)$$

$\Rightarrow$ WE CAN TEST THE OBSERVATION X

$$x \gtrless_{H_0}^{H_1} T$$

NOW:

$$\alpha = \int_T^\infty p_0(x)\,dx \Rightarrow \sigma\,\Phi^{-1}(1-\alpha)$$

$$\Rightarrow R_1 = \{x \,;\, x > \Phi^{-1}(1-\alpha)\}$$

THIS TEST IS UMP.

$$\beta(\theta) = \int_T^\infty p_1(x,\theta)\,dx$$

$$= 1 - \Phi\left(\frac{T-\theta}{\sigma}\right)$$

$$= 1 - \Phi\left[\Phi^{-1}(1-\alpha) - \frac{\theta}{\sigma}\right]$$

NOTE: MONOTONICALLY INCR.

72

11-5-75 (WED)

COMPOSITE HYPOTHESESE:

$H_0 : p(x;\theta) \qquad \theta \in \Theta_0$

$H_1 : p(X,\theta) \qquad \theta \in \Theta_1$

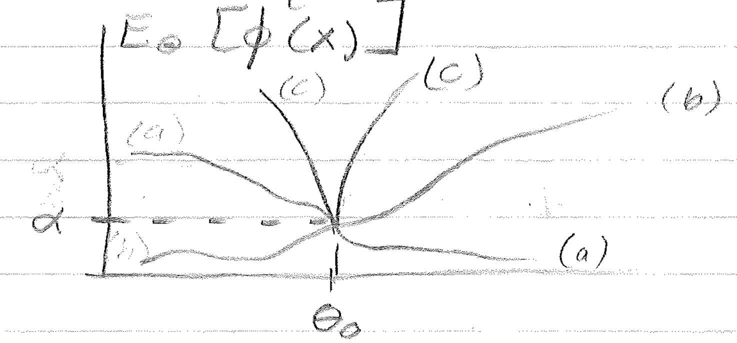
DEFINITION: THE SIZE, α, IS DEFINED AS $\alpha = \sup_{\theta \in \Theta_0} E_\theta[\phi(x)]$ (sup: LEAST UPPERBOUND)

$$E_\theta[\phi(x)] = \int \phi(x) p(x,\theta)\,dx$$

DEFINITION: A TEST ϕ_0 IS SAID TO BE UNIFORMLY MOST POWERFUL OF SIZE α FOR TESTING $H_0 : \theta \in \Theta_0$ AGAINST $H_1 : \theta \in \Theta_1$ IF ϕ_0 IS OF SIZE α, AND IF FOR ANY OTHER TEST IS AT MOST α:

$$E_\theta\{\phi_0(x)\} \geq E_\theta\{\phi(x)\} \quad \forall\, \theta \in \Theta_1$$

ASSUME THAT Θ_0 CONSISTS OF ONE POINT θ_0 (ie H_0 IS SIMPLE)



a AND b ARE NOT UMP

c IS AT LEAST A LOWER BOUND ON UMP.

IF c UPPER BOUNDS ALL OTHER TESTS, THEN IT IS U.M.P.

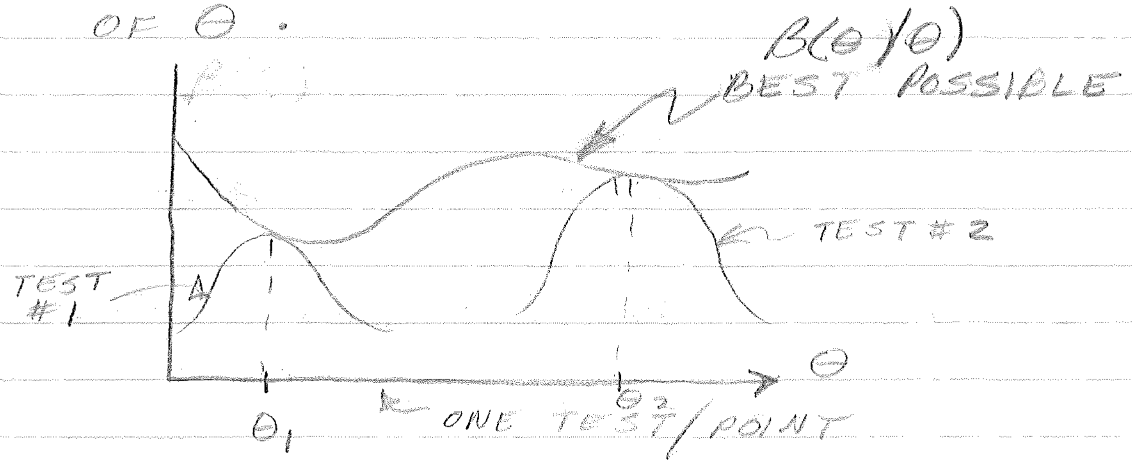
(CONTINUING:)

$H_0: \theta = \theta_0$

$H_1: \theta \in \Theta_1$

CONSIDER A FICTICIOUS "PERFECT MEASUREMENT TEST". i.e., DESIGN A N-P TEST OF SIZE α $\forall \theta \in \Theta_1$.

GRAPH THE POWER AS A FUNCTION OF θ.



THE BEST PERFORMANCE WOULD BE ACHEIVED IF THE ACTUAL TEST'S CURVES EQUALED THIS BOUND $\forall \theta \in \Theta_1$. SUCH TESTS ARE UMP. IN GENERAL, THE PERFORMANCE BOUND ($\beta(\theta)$) CAN BE REACHED FOR ANY PARTICULAR θ SIMPLY BY DESIGNING AN ORDINARY N-P TEST FOR THAT PARTICULAR θ. A UMP TEST MUST BE AS GOOD AS ANY OTHER TEST $\forall \theta$.

A UMP EXISTS <u>IFF</u> THE N-P TEST $\forall\,\theta$ CAN BE COMPLETELY DEFINED (INCLUDING THRESHOLD) WITHOUT KNOWLEDGE OF θ.

11-7-75 (FRI)

EXAMPLE:

$H_0: X = n$

$H_1: X = n + \theta$

NOISE $\sim N(0, \sigma^2)$, $\theta \neq 0$

$$\Lambda(x) = \exp\left[\frac{1}{2\sigma^2}(2x\theta - \theta^2)\right]$$

$$2x\theta - \theta^2 \gtrless^{H_1}_{H_0} T$$

$$x\theta \gtrless^{H_1}_{H_0} \tilde{T}$$ ⇐ AT THIS PT., GOTTA KNOW θ

∴ CANNOT DESIGN TEST WITHOUT KNOWLEDGE OF $\theta \Rightarrow$ UMP TEST DON'T EXIST

1. NOTE: SAME PROBLEM EXCEPT $\theta > 0$

$$\Rightarrow \quad x \gtrless^{H_1}_{H_0} \tilde{T}$$

$$\alpha = \int_{\hat{T}}^{\infty} p_0(x)\,dx \qquad \hat{T} = \sigma\Phi^{-1}(1-\alpha)$$

2. NOTE: SAME PROBLEM EXCEPT $\theta < 0$

$$\Rightarrow x \gtrless^{H_0}_{H_1} T$$ ← NOTE!

BOTH 1 & 2 ARE UMP.

$H_0 : X = n$

$H_1 : X = \theta + n$

$\theta > 0 \quad ; \quad \frac{\gamma}{2} e^{-\gamma|n|}$

$$\Lambda(x) = \exp\left[-\gamma|x-\theta| + \gamma|x|\right]$$

$$|x| - |x-\theta| \gtrless_{H_0}^{H_1} T$$

ABOUT AS FAR AS YOU CAN GO.

NO UMP. TEST EXISTS

DEFINITION: A REAL PARAMETER FAMILY OF DISTRIBUTIONS IS SAID TO HAVE MONOTONE LIKELIHOOD RATIO IF DENSITIES $P(X,\theta)$ EXIST $\ni$ WHEN $\theta_1 < \theta_2$, THE LIKELIHOOD RATIO $\frac{P(X;\theta_2)}{P(X,\theta_1)}$ IS A NONDECREASING FUNCTION OF X IN THE SET OF ITS EXISTANCE. (FERGUSON, p. 208).

THEOREM (FERGUSON, p. 210) IF THE DISTRIBUTION OF X HAS MONOTONE LIKELIHOOD RADIO, ANY TEST OF THE FORM

$$\phi(x) = \begin{cases} 1 & ; x > x_0 \\ p & x = x_0 \\ 0 & x < x_0 \end{cases}$$

IS UMP OF ITS SIZE $(\alpha \neq 0)$ FOR TESTING $H_0 : \theta \leq \theta_0$

AGAINST

$\theta > \theta_0$

$\forall \theta \in \Theta$.

CONSIDER A REAL PARAMETER FAMILY OF DISTRIBUTIONS. A DENSITY $p(x,\theta)$ COMES FROM A ONE PARAMETER EXPONENTIAL FAMILY IF IT CAN BE WRITTEN AS

$$p(x;\theta) = C(\theta)\,h(x)\,\exp\left[Q(\theta)\,T(x)\right]$$

ASSUME $p(x,\theta)$ COMES FROM A ONE PARAMETER EXPONENTIAL FAMILY. PICK $\theta_1 < \theta_2$.

$$\frac{p(x,\theta_2)}{p(x,\theta_1)} = \frac{C(\theta_2)}{C(\theta_1)} \exp\left\{\left[Q(\theta_2) - Q(\theta_1)\right] T(x)\right\}$$

IF $Q(\theta)$ AND $T(x)$ ARE NON-DECREASING, THEN THE LIKELIHOOD RATIO IS NONDECREASING $\Rightarrow p(x,\theta)$ HAS MONOTONE LIKELIHOOD RATIO.

EXAMPLE: $P(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\theta^2/2\sigma^2}\, e^{-x^2/2\sigma^2}\, e^{\frac{\theta x}{\sigma^2}}$$

$\therefore P(x;\theta)$ IS A 1 PAR. EXPON. FAMILY

CONSIDER

$$H_0;\ x = \theta + n;\ \theta \le 0$$
$$H_1;\ x = \theta + n \quad \theta > 0$$

RECALL

$$\alpha = \sup_{\theta \le 0} \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}} dx$$

$$= \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} dx$$

$x_0 = \sigma\,\Phi^{-1}(1-\alpha)$ (CONT)

TEST IS

$$X \underset{H_0}{\overset{H_1}{\gtrless}} \sigma \Phi^{-1}(1-\alpha)$$

ANY (MEASUREABLE) TRANSFORMATION ON $\underline{X}$ IS CALLED A STATISTIC OF $\underline{X}$.

11-10-75 (MON)

DEFINITION: LET X DENOTE A RANDOM VARIABLE (OR VECTOR) WHOSE DISTRIBUTION DEPENDS ON A PARAMETER $\theta \in \Theta$. A REAL (OR VECTORED) VALUED FUNTION T OF $\underline{X}$ IS SAID TO BE SUFFICIENT FOR θ IF THE CONDITIONAL DISTRIBUTION OF $\underline{X}$ GIVEN $T=t$, IS INDEPENDENT OF θ. (FROM FERGUSON, p. 113)

LET T BE A SUFFICIENT STATISTIC, THEN IF ONE IS PERMITTED TO OBSERVE ONLY T INSTEAD OF $\underline{X}$, THIS DOES NOT RESTRICT THE CLASS OF AVAILABLE DECISION PROCEDURES (LEYMANN, p. 18)

FACTORIZATION THEOREM:
T IS A SUFFICIENT STATISTIC 'IFF' $\exists$ NON-NEGATIVE (MEASURABLE) g AND h $\ni$

$$p(x,\theta) = g(t,\theta)h(x)$$

(PROVED IN LEHMANN, pp. 49-50) HAIRY

RECALL, $p(x;\theta) = C(\theta)\, h(x)\, e^{Q(\theta)T(\theta)}$
(ONE PARAMETER EXPON. FAMILY)

IT FOLLOWS THAT $T(x)$ IS A SUFFICIENT STATISTIC IN THE EXPONENTIAL FAMILY.

NOTE: GIVEN AN ARBITRARY ONE PARAMETER EXPONENTIAL FAMILY OF DISTRIBUTIONS, THE NONDECREASINGNESS OF $T(x)$ CAN BE OBTAINED BY A CHANGE OF VARIABLE $y = T(x)$, IF NECESSARY.

PROPOSITION: $H_0: p(x;\theta)$; $\theta \le \theta_0$
$H_1: p(x;\theta)$; $\theta > \theta_0$

$$p(x;\theta) = K(\theta)\, h(x)\, \exp[Q(\theta)T(x)]$$

$\Rightarrow Q(\theta)$ IS NONDECREASING

ASSUME WE GOT K iid SAMPLES.

$$\Rightarrow \text{TEST IS } \sum_{1}^{K} T(x_k) \underset{H_0}{\overset{H_1}{\gtrless}} X_0$$

(SIMILAR IN FORM TO PREVIOUS EFFORTS, BUT NOTE DIFFERENT HYPOTHESIS)

HAND WAVING

$$p(x_1, x_2, \ldots, x_K, \theta) = p(x_1, \theta)\, p(x_2, \theta) \ldots p(x_K, \theta)$$
$$= [C(\theta)]^K\, [h(x_1)\, h(x_2) \ldots h(x_K)]$$
$$\times \Big[e^{Q(\theta) \underbrace{\sum_{k=1} T(x_k)}_{=Y}}\Big]$$

(IMPORTANT TO UNDERSTAND)

EXAMPLE: WE OBSERVE $\cancel{K}$ iid EVENTS THAT ARE EIGTHER SUCCESS OR FAILURE. WE WANNA DECIDE IF THE PROBABILITY OF FAILURE θ IS $\leq p$, AND WE WANT PROBABILITY OF INCORRECTLY ANNOUNCING THAT $\theta > p$ TO BE $\leq \alpha$. LET N BE THE RANDOM VARIABLE THAT REPRESENTS k iid OBSERVATIONS.

$$p(N = n, \theta) = \binom{K}{n} \theta^n (1-\theta)^{K-n}$$

THIS IS A BINOMIALLY DISTRIBUTED R.V. $H_0 : \theta \leq p$

$\theta > p$

$$p(N = n, \theta) = \binom{K}{n} (1-\theta)^K \left(\frac{\theta}{1-\theta}\right)^n$$

$$= \binom{K}{n} (1-\theta)^K e^{n \ln \theta/1-\theta}$$

NOTE: $\ln \theta/1-\theta$ IS NONDECREASING

$\Rightarrow$ MONOTONE LIKLIHOOD RATO, THE TEST IS

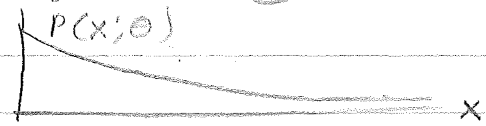
$$\phi \begin{cases} 1 & ; n > n_0 \\ \rho & ; n = n_0 \\ 0 & ; n < n_0 \end{cases}$$

$$\alpha = \sup_{\theta \leq p} E_\theta[\phi(N)] =$$
$$= \sup_{\theta \leq p} \rho \binom{K}{n_0} \theta^{n_0} (1-\theta)^{K-n_0}$$
$$+ \sum_{n=n_0+1}^{K} \binom{K}{n} \theta^n (1-\theta)^{K-n}$$

11-12-75 (WED)

EXAMPLE: LIFETIME TESTING, OR TIME OF OCCURANCE

$$p(x;\theta) = \theta e^{-\theta x}, \quad x \geq 0, \quad \theta > 0$$



$H_0: \theta \leq t$

$H_1: \theta > t$

ASSUME K iid REALIZATION.

$x_1, x_2, \ldots, x_K$

TEST IS $\sum_{k=1}^{K} x_k \gtrless^{H_0}_{H_1} x_0$

CHOOSE $x_0 \ni \alpha = \sup_{\theta \leq t} P_\theta\left[\sum_{k=1}^{K} x_k < x_0\right]$

EXAMPLE:

$H_0: \theta \leq \theta_0$

$H_1: \theta > \theta_0$

$$p(x,\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}, \quad \theta > 0$$

K iid REALIZATIONS

TEST IS: $\sum_{k=1}^{K} x_k^2 \gtrless^{H_1}_{H_0} X_0^2$

UMP TEST (THE END)

$H_0: p_0(x)$
$H_1: p_1(x, \theta)$

I. IF WE KNOW θ, THEN THE TEST IS

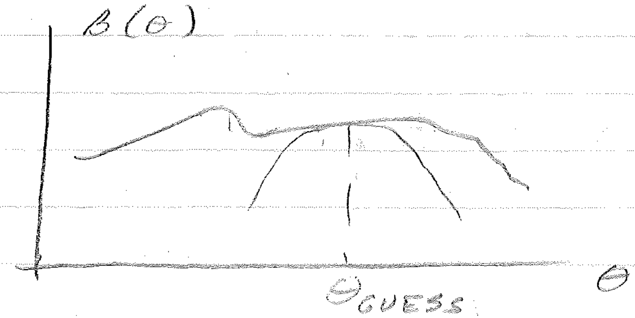
$$\Lambda(x,\theta) = \frac{p_1(x,\theta)}{p_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$

II. IF WE DON'T KNOW θ:

A. SOLVE FOR LIKELIHOOD RATIO
B. SPECIFY α
C. FIND ACCEPTANCE REGION (DEFINED IMPLICITLY)
D. IF IT IS INDEPENDENT OF θ,
YOU GOTTA UMP TEST

WE ARE AT THE POINT UMP DON'T EXIST.

RECALL THE PERFECT MEASUREMENT BOUND ON THE PWR. OF THE TEST.
i.e. DESIGN A N-P TEST OF SIZE α $\forall \theta$.



THIS MIGHT SUGGEST THE FOLLOWING.
① ESTIMATE θ BASED ON OBSERVATION.
CALL THE ESTIMATE $\hat{\theta}(x)$
② THEN USE THE TEST

$$\frac{p_1(x, \hat{\theta}(x))}{p_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$

11-14-75 (FRI)

MORE GENERAL CASE

$H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_1$

LET $\hat{\Theta}_0(X)$ BE AN ESTIMATE OF θ ASSUMING H_0 IS TRUE. LET $\hat{\Theta}_1$ BE AN ESTIMATE OF Θ ASSUMING H_1 IS TRUE. THEN USE

$$\frac{P_1[X, \hat{\Theta}_1(X)]}{P_0(X, \hat{\Theta}_0(X)]} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$

ESTIMATION OF SIGNAL PARAMETERS

DENSITY INDEXED BY AN UNKNOWN

$$\theta \in \Theta \Longrightarrow p(x, \theta)$$

WE OBSERVE $\underline{X}$ AND WANNA ESTIMATE θ. CALL THE ESTIMATE $\hat{\theta}(x)$.

$$E[\hat{\theta}(\underline{X})] = \int \hat{\theta}(x)\, p(x, \theta)\, dx$$

IF $E[\hat{\theta}(x)] \in \Theta\ \forall \theta$ THEN $\hat{\theta}(x)$ IS SAID TO BE UNBIASED.

IF $E[\hat{\theta}(\underline{X})] = \theta + B$, THE ESTIMATE HAS A KNOWN BIAS.

NOTE: CAN ALWAYS OBTAIN AN UNBIASED ESTIMATE BY SUBTRACTING B, i.e. USE $\hat{\Theta}(x) - B$.

IF $E[\hat{\theta}(\underline{X})] = \theta + B(\theta)$ THEN THE ESTIMATE HAS AN UNKNOWN BIAS: CAN'T SIMPLY SUBTRACT IT OUT.

NOTE: EVEN AN UNBIASED ESTIMATE MAY LEAD TO A BAD RESULT ON A PARTICULAR TRIAL.

WE WOULD LIKE THE DENSITY OF $\hat{\theta}(x)$ TO BE CONCENTRATED AROUND θ. WE WOULD LIKE $\hat{\theta}(x)$ TO BE UNBIASED AND TO HAVE A VERY SMALL VARIANCE. ASSUME $\hat{\theta}(x)$ IS UNBIASED. (i.e., $E[\hat{\theta}(x)] = \theta$). NOW

$$\mathrm{var}[\hat{\theta}(x)] = E[\{\hat{\theta}(x - \theta)^2] = 0$$

RECALL CHEBYCHEV'S INEQUALITY:

$$P[|\hat{\theta}(x) - \theta| > \lambda] \leq \frac{\mathrm{var}[\hat{\theta}(x)]}{\lambda^2}$$

LET $\hat{\theta}(x)$ BE AN UNBIASED ESTIMATE OF θ. (ASSUME ALL MATHEMATICAL OPERATIONS ARE JUSTIFIED).

FIRST OFF, $E[\hat{\theta}(x) - \theta(x)]$

$$= \int [\hat{\theta}(x) - \theta] p(x,\theta)\, dx$$

TAKE DERIVATIVE W/ RESPECT TO θ

$$0 = -\int p(x;\theta)\,dx + \int [\hat{\theta}(x) - \theta) \frac{\delta}{\delta\theta} p(x;\theta)\,dx$$

$$\Rightarrow \int [\hat{\theta}(x) - \theta] \frac{\delta}{\delta\theta} p(x;\theta)\,dx = 1$$

NOTE: $\frac{\delta}{\delta\theta} p(x;\theta) = p(x;\theta) \frac{d}{d\theta} \ln p(x,\theta)$

$$\Rightarrow \int [\hat{\theta}(x) - \theta] p(x,\theta) \frac{\delta}{\delta\theta} \ln p(x,\theta)\, dx = 1$$

$$1 = \int \left(\sqrt{p(x;\theta)} \frac{\delta}{\delta\theta} \ln p(x;\theta)\right) \cdot$$
$$\cdot \sqrt{p(x;\theta)}\, [\hat{\theta}(x) - \theta]\, dx$$

(CONT.)

USE SCHWARZ'S INEQUALITY

$$\left|\int f(t)g(t)dt\right|^2 \leq \int |f(t)|^2 dt \int |g(t)|^2 dt$$

THUS

$$1 \leq \underbrace{\int p(x;\theta)\left[\frac{\delta}{\delta\theta} p(x;\theta)\right]^2 d\theta}_{E\left[\left(\frac{\delta}{\delta\theta} p(x;\theta)\right)^2\right]} \cdot \underbrace{\int p(x;\theta)\left[\hat{\theta}(x)-\theta\right]^2 dx}_{var\left[\hat{\theta}(x)\right]}$$

THUS

$$var\left[\hat{\theta}(x)\right] \geq \frac{1}{E\left\{\left(\frac{\delta}{\delta\theta} \ln p(x,\theta)\right)^2\right\}}$$

THIS IS CRAMÉR - RAO INEQUALITY

11-17-75 (MON)

ALTERNATE FORM

NOTE : $\int p(x,\theta)dx = 1$

$$\int \frac{\delta}{\delta\theta} p(x;\theta)dx = 0$$

$$= \int p(x;\theta) \frac{\delta}{\delta\theta} \ln p(x;\theta) dx$$

TAKE $\frac{\delta}{\delta\theta}$ AGAIN GIVES

$$0 = \int p(x;\theta)\left[\frac{\delta}{\delta\theta} \ln p(x;\theta)\right]^2 + \int p(x;\theta) \frac{\delta^2}{\delta\theta^2} p(x;\theta) d\theta$$

$$\Rightarrow E\left[\left(\frac{\delta}{\delta\theta} \ln p(x;\theta)\right)^2\right] = -E\left[\frac{\delta^2}{\delta\theta^2} \ln p(x;\theta)\right]$$

$$\therefore \operatorname{var}(\hat{\theta}(x)) \geq \frac{1}{E[(\frac{\delta}{\delta\theta} \ln p(x,\theta))^2]}$$

$$= \frac{-1}{E[\delta^2/d\theta^2 \ln p(x;\theta)]}$$

NOTE: ANY UNBIASED ESTIMATE OF θ MUST HAVE A VARIANCE GREATER THAN A CERTAIN NUMBER.

IF THE ESTIMATE IS BIASED, WE CAN SHOW THAT

$$E[\{\hat{\theta}(x) - \theta\}]^2 \geq \frac{[1 + \frac{\delta B(\theta)}{\delta\theta}]^2}{E\{[\frac{\delta}{\delta\theta} \ln p(x;\theta)]^2\}}$$

NOTE: SIMILAR BOUNDS ARE AVAILABLE FOR THE CASE WHERE θ IS MULTIDIMENSIONAL.

EXAMPLE: $p(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$

$$\ln p(x,\theta) = -\ln\sqrt{2\pi}\,\sigma - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\delta}{\delta\theta} \ln p(x,\theta) = \frac{x-\theta}{\sigma^2}$$

$$[\tfrac{\delta}{\delta x} \ln p(x,\theta)]^2 = \frac{(x-\theta)^2}{\sigma^4}$$

$$E[(\tfrac{d}{dx} \ln p(x,\theta))]^2 = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}$$

$$\Rightarrow E[\{\hat{\theta}(x) - \theta\}^2] \geq \sigma^2$$

WHERE $\hat{\theta}(x)$ IS ANY UNBIASED ESTIMATE OF θ.

(CONTINUED)

ASSUME THAT WE CHOOSE $\hat{\theta}(x) = \underline{X}$

THEN $E[\hat{\theta}(x)] = E[\underline{X}] = \theta$

$E\{[\hat{\theta} - \theta)^2] = E[(x - \theta)^2] = \sigma^2$

IN THIS CASE, THIS ESTIMATE IS UNBIASED & IT ACHEIVES THE C.R. BOUND.

EXAMPLE: $p(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}$, $\theta > 0$

$\ln(p(x,\theta)) = -\frac{1}{2}\ln 2\pi\theta - x^2/2\theta$

$\frac{\delta}{\delta\theta} p(x,\theta) = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2}$

$= \frac{x^2 - \theta}{2\theta^2}$

$[\frac{\delta}{\delta\theta} p(x,\theta)]^2 = \frac{x^4 - 2\theta x^2 + \theta^2}{4\theta^4}$

$E[\{\frac{\delta}{\delta\theta} p(x,\theta)\}^2] = \frac{3\theta^2 - 2\theta^2 + \theta^2}{4\theta^4} = \frac{1}{2\theta^2}$

$(\text{SINCE } x \sim N(0,\sigma^2) \Rightarrow E[x^4] = 3\sigma^4)$

$\therefore E[\{\hat{\theta}(x) - \theta\}]^2 \geq 2\theta^2$

$\ni \hat{\theta}$ IS ANY UNBIASED ESTIMATE OF THETA.

DEFINITION: AN UNBIASED ESTIMAE THAT ACHEIVES THE C-R LOWER BOUND IS CALLED AN EFFICIENT ESTIMATE

LET $\hat{\theta}(x)$ BE AN UNBIASED ESTIMATE OF θ. RECALL THE PROOF OF THE CRAMER-RAO INEQUALITY. BASED ON THE SCHWARZ INEQUALITY. EQUALITY HOLDS IFF THE TWO TERMS IN THE INNER PRODUCT ARE RELATED BY A SCALE FACTOR

$\Rightarrow \hat{\theta}(x)$ IS EFFICIENT IFF

$$\sqrt{p(x,\theta)}\,\frac{\delta}{\delta\theta}\ln p(x,\theta) = K(\theta)\sqrt{p(x,\theta)}\,[\hat{\theta}(x)-\theta]$$

WHERE $K(\theta)$ IS IND. OF X.

$$\Rightarrow \frac{\delta}{\delta\theta}\ln p(x,\theta) = K(\theta)[\hat{\theta}-\theta]$$

$\therefore$ THE UNBIASED ESTIMATE, $\hat{\theta}(x)$, IS EFFICIENT IFF

$$\frac{\delta}{\delta\theta}\ln p(x,\theta) = K(\theta)[\hat{\theta}-\theta]$$

EXAMPLE: $p(x,\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$

$$\frac{\delta}{\delta x}\ln p(x,\theta) = \frac{x-\theta}{\sigma^2} \Rightarrow \hat{\theta}(x) = X$$

$$E[\hat{\theta}(x)] = \theta$$

EXAMPLE: $p(x,\theta) = \frac{1}{\sqrt{2\pi\theta}}\, e^{-x^2/2\theta}$

$$\frac{\delta}{\delta\theta}\ln p(x,\theta) = \frac{x^2-\theta}{2\theta^2}$$

$$E[X^2] = \theta$$

$$\hat{\theta}(x) = x^2$$

EFFICIENT ESTIMATE HERE ALSO.

11-18-75 (WED)

MAXIMUM LIKELIHOOD ESTIMATION

$p(x;\theta)$

CHOOSE THE VALUE OF θ THAT MAXIMIZES $p(x,\theta)$. ie, WE LOOK AT $p(x;\theta)$ AS A FUNCTION OF θ, SAY $L(\theta)$, THE LIKELIHOOD FUNCTION. THEN MAXIMIZE $L(\theta)$. THE SOLUTION $\hat{\theta}$ IS THE MAXIMUM LIKELIHOOD EXTIMATE OF θ.

WE CHOOSE THAT VALUE OF θ WHICH MAXIMIZES THE PROBABILITY THAT THE OBSERVATION IS IN A SMALL NEIGHBORHOOD OF X, ie $p(x,\theta)dx$, ie WE CHOOSE θ WHICH MAKES THE "AFTER THE FACT" DENSITY MAXIMUM. $L(\theta)=$ MAX.

IF DERIVITIVES EXIST,

$$\frac{\delta L(\theta)}{\delta\theta} = \frac{\delta p(x;\theta)}{\delta\theta} = 0$$

NOTE: MAXIMIZING $L(\theta)$ IS EQUIVALENT TO MAXIMIZING $\ln L(\theta)$.

$$\Rightarrow \frac{\delta}{\delta\theta}\ln L(\theta) = 0 = \frac{\delta}{\delta\theta}\ln p(x;\theta)$$

$\frac{\delta}{\delta\theta}\ln p(x;\theta)=0 \Leftarrow$ THIS IS CALLED THE LIKELIHOOD EQUATION. (NOTE: THIS IS ASSUMING THAT THE MAXIMUM IS INTERIOR TO THE RANGE OF θ)

FINDING THE ROOTS OF $\frac{\delta}{\delta\theta} p(x;\theta)\big|_{\hat{\theta}_{me}} = 0$
IS EQUIVALENT TO FINDING ROOTS
OF $\frac{\delta}{\delta\theta} \ln p(x,\theta)\big|_{\hat{\theta}_{me}} = 0$, $\hat{\theta}_{me}$ = MAXIMUM LIKE
RECALL
IF $\hat{\theta}$ IS AN EFFICIENT ESTIMATE, THEN

$$\frac{\delta}{\delta\theta} \ln p(x;\theta) = K(\theta)\left[\hat{\theta} - \theta\right]$$

ASSUME $\hat{\theta}$ IS AN EFFICIENT ESTIMATOR.
NOW $0 = \frac{\delta}{\delta\theta} \ln p(x;\theta)\big|_{\hat{\theta}_{me}} = K(\theta)\left[\hat{\theta} - \hat{\theta}_{me}\right]$
GIVES TWO CASES:

1. $K(\hat{\theta}_{me}) = 0$
2. $\hat{\theta}(x) = \hat{\theta}_{me}(x)$

CONSIDER CASE 1: FOR SOME NUMBER α, $K(\alpha) = 0$. THIS IMPLIES THAT THE MAXIMUM LIKELIHOOD ESTIMATE, α, IS INDEPENDENDENT OF THE DATA.
$\Rightarrow$ REJECT CASE 1.

$\therefore \hat{\theta}(x) = \hat{\theta}_{me}(x)$

WE CONCLUDE: IF AN EFFICIENT ESTIMATE EXISTS, IT EQUALS THE MAXIMUM LIKELIHOOD ESTIMATE.

NOTE: IF AN EFFICIENT ESTIMATE DOES NOT EXIST, ie $\frac{\delta}{\delta\theta} \ln p(x,\theta)$ CANNOT BE PUT INTO THE FORM $K(\theta)[\hat{\theta}-\theta]$, THEN WE DO NOT KNOW HOW GOOD $\hat{\theta}_{ml}(x)$ IS. FURTHERMORE WE DO NOT KNOW HOW CLOSE THE VARIANCE OF ANY ESTIMATE WILL APPROACH THE CR BOUND.

11-21-75 (FRI)

NOTE: THE MAXIMUM LIKLIHOOD PRINCIPLE IS NOT BASED ON ANY CLEARLY DEFINED OPTIMUM CONSIDERATIONS.

PROPERTIES

1. INVARIANT: IF $\hat{\theta}_{ml}$ IS THE MAXIMUM LIKELIHOOD ESTIMATE OF θ, AND $f(\theta)$ IS A MONOTONE FUNCTION OF θ, THEN $f(\hat{\theta}_{ml})$ IS THE MAXIMUM LIKELIHOOD ESTIMATE OF $f(\theta)$

2. LET $X_1, X_2, \ldots, X_n$ BE INDEPENDENT REALIZATIONS.

$$\hat{\theta}_{ml} = \hat{\theta}_{ml}(X_1, X_2, \ldots, X_n)$$

UNDER REASONABLY GENERAL CONDITIONS, THE FOLLOWING MAY BE PROVED (SEE CRAMER: MATHEMATICAL METHODS OF STATISTICS, pp. 500-504.

- CONSISTANCY: $\hat{\Theta}_{ml}$ IS A CONSISTENT ESTIMATE OF Θ.

  ie $\hat{\Theta}_{ml} \xrightarrow{P} \Theta_0$, Θ_0 = TRUE VALUE OF Θ

  ie $\forall \epsilon > 0$, $\lim_{n\to\infty} P[\,|\hat{\Theta}_{ml} - \Theta_0| > \epsilon\,] = 0$

- ASYMPTOTICALLY UNBIASED

  $$\lim_{n\to\infty} E[\hat{\Theta}_{ml}] = \Theta_0$$

- ASYMPTOTICALLY EFFICIENT

  $$\lim_{n\to\infty} E[(\hat{\Theta}_{ml} - \Theta_0)^2] = \frac{1}{E([\frac{\delta}{\delta\Theta_0} \ln p(x,\Theta_0)]^2)}$$

- ASYMPTOTICALLY GAUSSIAN: THE PROPABILITY DISTRIBUTION FUNCTION OF $\hat{\Theta}_{ml}$ CONVERSES TO A GAUSSIAN DISTRIBUTION FUNCTION.

COMMENT: LET $T(x)$ BE A SUFFICIENT STATISTIC FOR Θ. THEN THE MAXIMUM LIKELIHOOD ESTIMATE MUST BE A FUNCTION OF T. ie. $p(x,\Theta) = g(T,\Theta)\,h(x)$

$$\frac{\delta}{\delta\Theta} \ln p(x;\Theta) = \frac{\delta}{\delta\Theta} \ln g(T,\Theta)$$

CONSIDER THE DETECTION PROBLEM

$H_0 : p_0(x,\theta) \quad , \theta \in \Theta_0$

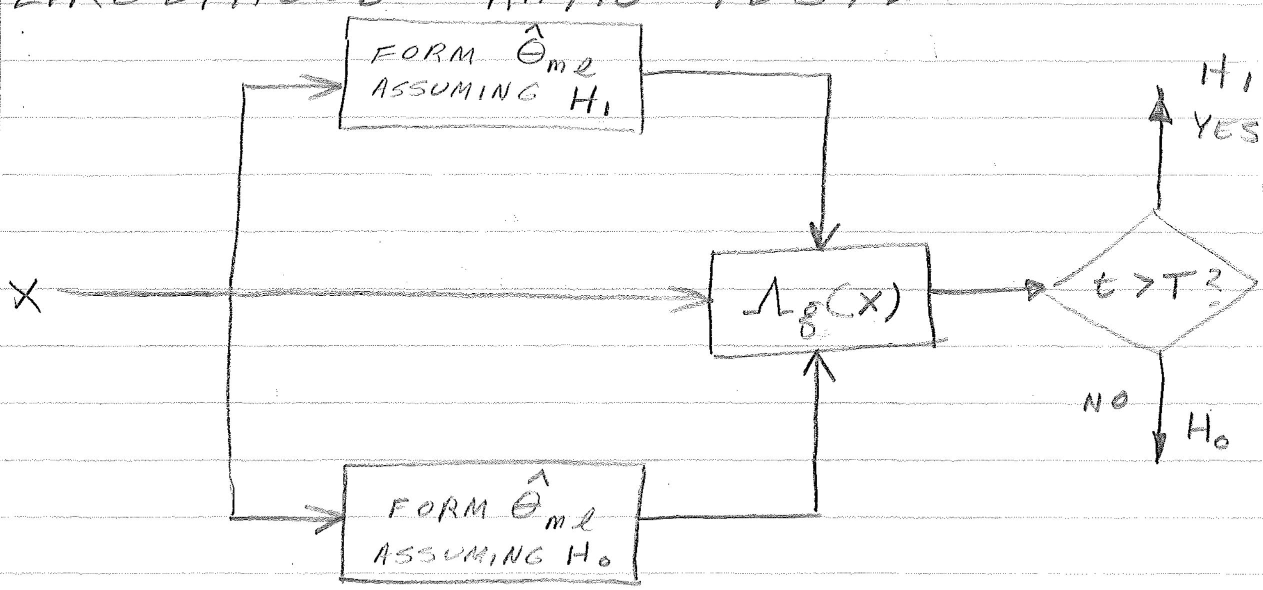
$H_1 : p_1(x;\theta) \quad ; \theta \in \Theta_1$

(SHOULD FIRST LOOK FOR UMP TEST)

ASSUME THAT A UMP TEST DOESN'T EXIST. WE WILL USE A MAXIMUM LIKELIHOOD PROCEDURE.

$$\Lambda_g(x) = \frac{\max_{\theta \in \Theta_1} p_1(x,\theta)}{\max_{\theta \in \Theta_0} p_0(x,\theta)} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$

THIS IS CALLED A GENERALIZED LIKELIHOOD RATIO TEST.



EXAMPLE: $H_0 : x = n$

$H_1 : x = \theta + n$

NOISE $\sim N(0, \sigma^2)$

(IF WE KNOW IF θ IS EITHER POSITIVE OR NEGATIVE, WE USE A UMP TEST. ASSUME THAT WE DON'T KNOW THE POLARITY OF θ

ASSUME WE TAKE K SAMPLES, AND CHOOSE TO DO MAXIMUM LIKLIHOOD.

UNDER H_0

$$p_0(x_1, x_2, \ldots, x_K) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^K \exp -\frac{1}{2\sigma^2}\sum_{K=1}^{K} x_K^2$$

UNDER H_1

$$p_1(x_1, x_2, \ldots, x_K) = \prod_{K=1}^{K} \exp -\frac{1}{2\sigma^2}\sum_{K=1}^{K}(x_K - \theta)^2$$

TO FIND $\hat{\theta}_{me}$

$$\frac{\delta}{\delta\theta} \ln p(x_1, x_2, \ldots x_K; \theta) = 0$$

GIVES $\Rightarrow \sum_{K=1}^{K}(x_K - \theta) = 0$

OR $\sum_{K=1}^{K} x_K = \sum_{K=1}^{K} \theta = K\theta$

$$\Rightarrow \hat{\theta}_{me}(x_1, x_2, \ldots, x_K) = \frac{1}{K}\sum_{K=1}^{K} x_K$$

$$\Lambda_g(x_1, x_2, \ldots x_K) = \frac{\exp -\frac{1}{2\sigma^2}\sum_{K=1}^{K}(x_K - \hat{\theta}_{me})^2}{\exp -\frac{1}{2\sigma^2}\sum_{K=1}^{K} x_K^2}$$

$$\ln \Lambda_g = \frac{1}{\sigma^2}\sum_{K=1}^{K}\left[x_K\hat{\theta}_{me} - \frac{1}{2}(\hat{\theta}_{me})^2\right]$$

$$\sigma^2 \ln \Lambda_g = \sum_{K=1}^{K} x_K\left(\frac{1}{K}\sum_{j=1}^{K} x_j\right) - \frac{1}{2}\left(\frac{1}{K}\sum_{j=1}^{K} x_j\right)^2$$

$$= \sum_{K=1}^{K}\sum_{j=1}^{K} x_K x_j - \frac{1}{2K}\sum_{K=1}^{K} x_K^2$$

$$= \frac{1}{2K}\left(\sum_{K=1}^{K} x_K\right)^2$$

$$\Rightarrow 2K\sigma^2 \ln \Lambda_g = \left(\sum_{K=1}^{K} x_K\right)^2$$

TEST IS

$$\left(\sum_{K=1}^{K} X_K\right)^2 \underset{H_0}{\overset{H_1}{\gtrless}} T^2$$

11-24-75 (MON)

$$\left(\sum_{K=1}^{K} X_K\right)^2 = (K\hat{\theta}_{me})^2 \underset{H_0}{\overset{H_1}{\gtrless}} T^2$$

UNDER H_0 : $\sum_{K=1}^{K} X_K \sim N(0, K\sigma^2)$

UNDER H_1 : $\sum_{K=1}^{K} X_K \sim N(K\theta, K\sigma^2)$

NOW: $\alpha = P_0\left[\left(\sum_{K=1}^{K} X_K\right)^2 > T^2\right]$ ← NOTE: NOT OPTIMAL

$$= 1 - P_0\left[-T \leq \sum_{K=1}^{K} X_K \leq T\right]$$

$$= 1 - \int_{-T}^{T} \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{x^2}{2K\sigma^2}} dx$$

$$= 1 - \int_{-\frac{T}{\sqrt{K}\sigma}}^{\frac{T}{\sqrt{K}\sigma}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= 2\int_{-\infty}^{-T/\sqrt{K}\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= 2\,\Phi\left[\frac{-T}{\sqrt{K}\sigma}\right]$$

$$\Rightarrow T = -\sqrt{K}\,\sigma\,\Phi^{-1}\left(\frac{\alpha}{2}\right) > 0$$

$$T^2 = K\sigma^2\left[\Phi^{-1}\left(\frac{\alpha}{2}\right)\right]^2$$

NOW:

$$\beta_{ml} = P_1\left[\sum_{K=1}^{K} x > T^2\right]$$

$$= 1 - P_1\left[-T \le \sum_{K=1}^{K} X_K \le T\right]$$

$$= 1 - \int_{-T}^{T} \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-(u-K\theta)^2/K\sigma^2} du$$

$$= 1 - \int_{\frac{-T-K\theta}{\sqrt{K}\,\sigma}}^{\frac{T-K\theta}{\sqrt{K}\,\sigma}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= 1 + \Phi\left[\frac{-T-K\theta}{\sqrt{K}\sigma}\right] - \Phi\left[\frac{T-K\theta}{\sqrt{K}\sigma}\right]$$

PLUG IN EXPRESSION FOR T:

$$\beta_{ml} = 1 + \Phi\left[\Phi^{-1}\left(\frac{\alpha}{2}\right) - \frac{\theta}{\sigma}\sqrt{K}\right]$$

$$- \Phi\left[-\Phi^{-1}\left(\frac{\alpha}{2}\right) - \frac{\theta}{\sigma}\sqrt{K}\right] \Leftarrow \text{EVEN IN } \theta$$

$\beta_{ml}(\theta) = \beta_{ml}(-\theta)$ i.e. $f(|\theta|)$

NOTE: $\lim_{K\to\infty} \beta = 1$

TO FIND OUT HOW GOOD THIS IS, ASSUME WE KNOW θ (SAY $\theta > 0$).

GIVES $T = \sqrt{K}\,\sigma\, \Phi^{-1}(1-\alpha)$

AND TEST IS $\sum_{K=1}^{K} X_K \gtrless_{H_0}^{H_1} T$

AND $\beta_{ml} = \int_T^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-(x-K\theta)^2/2K\sigma^2} dx$

$$= \int_{\frac{T-K\theta}{\sqrt{K}\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = 1 - \Phi\left[\frac{T-K\theta}{\sqrt{K}\,\sigma}\right]$$

$$= 1 - \Phi\left[\Phi^{-1}(1-\alpha) - \frac{|\theta|}{\sigma}\sqrt{K}\right]$$

MAY WRITE

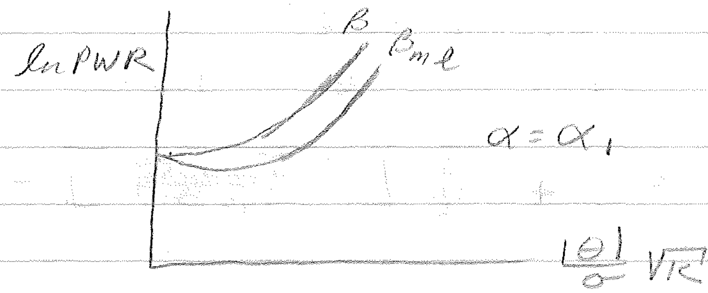
GOING THRU THE TEST WITH $\theta < 0$

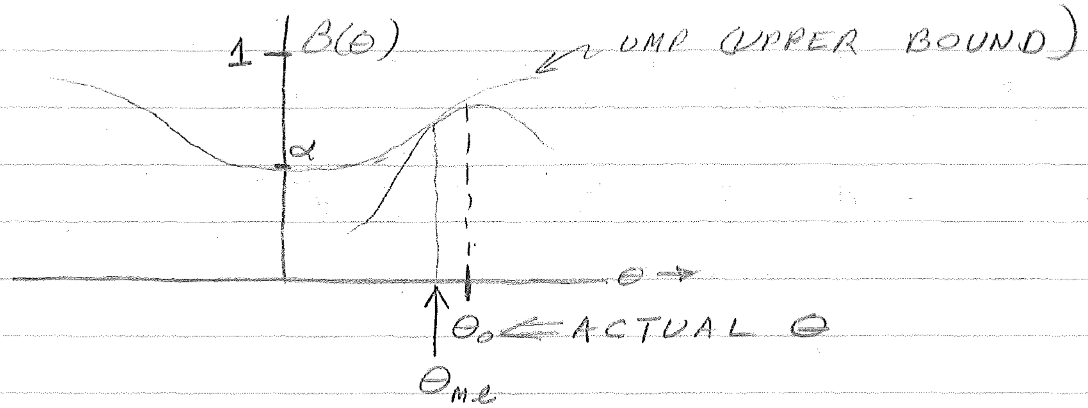
FOR $\sqrt{K}\,|\theta|/\sigma$ LARGE, THIS β GETS REALLY CLOSE TO β_{ml}.

e.g. ① $\alpha = 10^{-2}$, $\frac{\sqrt{K}\,|\theta|}{\sigma} = 5$, THEN BOTH β AND β_{ml} ARE GREATER THAN 0.99

② $\alpha = 10^{-8}$, $\sqrt{K}\,|\theta|/\sigma = 8.5$ THEN BOTH β AND β_{ml} ARE GREATER THAN 0.99

ROUGHLY:





11-26-75 (WED)

(TEST NEXT WEDNESDAY)

EXAMPLE: CONTINUOUS TIME DETECTION IN GAUSSIAN NOISE. UNKNOWN SIGNAL AMPLITUDE

$$H_0: x(t) = n(t)$$

$$H_1: x(t) = \Theta s(t) + n(t) \qquad ; \; t \in [a, b]$$

ZERO MEAN GAUSSIAN NOISE WITH CONTINUOUS POSITIVE-DEFINATE AUTOCORRELATION FUNCTION $R(t_1, t_2)$. LET $q(t)$ BE DEFINED BY

$$s(t_1) = \int_a^b R(t_1, t_2)\, q(t_2)\, dt_2$$

THE LOG LIKLIHOOD RATIO IS

$$\ln \Lambda[x(t), \Theta] = \Theta \int_a^b x(t) q(t)\, dt - \frac{1}{2}\Theta^2 \int_a^b s(t) q(t)\, dt$$

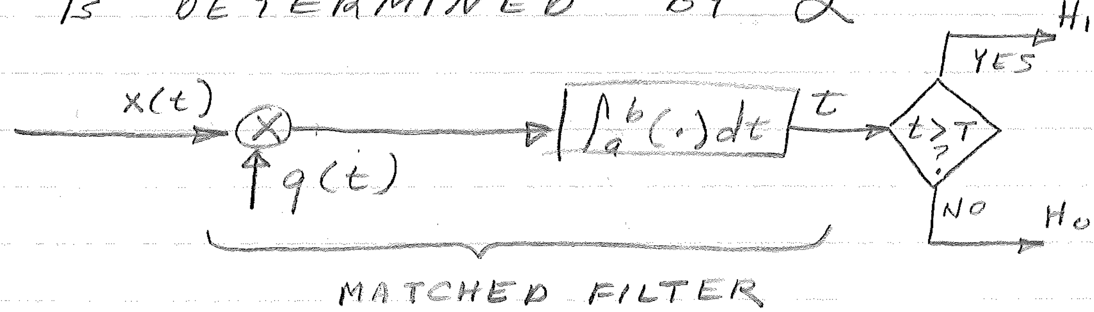
IF THE SIGN OF Θ IS KNOWN, WE HAVE A UMP TEST.

e.g. ASSUME $\Theta > 0$

THEN TEST IS:

$$\int_a^b q(t) x(t)\, dt \gtrless_{H_0}^{H_1} T$$

T IS DETERMINED BY α



ASSUME THAT THE POLARITY OF θ IS UNKNOWN. THEN NO UMP TEST EXISTS. CHOOSE θ SUCH THAT THE LOG LIKLIHOOD RATIO IS MAXIMIZED. (THIS MAXIMIZES $p_1(x;\theta)$.

IF $f(\theta) = a\theta - \frac{1}{2} b\theta^2$

$$f'(\theta) = a - b\theta$$

$$f(a/b) \geq f(\theta) \;\forall \theta$$

$$\Rightarrow \hat{\theta}_{me} = \frac{\int_a^b x(t)\,q(t)\,dt}{\int_a^b s(t)\,q(t)\,dt}$$

NOTE: $E_1[\hat{\theta}_{me}] = \dfrac{E\left[\int_a^b [\theta s(t) + n(t)]\,q(t)\,dt\right]}{\int_a^b s(t)\,q(t)\,dt}$

$= \theta \Leftarrow$ $\hat{\theta}_{me}$ IS AN UNBIASED ESTIMATE

NOW:

$$\frac{\delta}{\delta\theta} \ln \Lambda[x(t),\theta] = \int_a^b x(t)q(t)dt - \theta\int_a^b s(t)q(t)dt$$

$$= \frac{\delta}{\delta\theta} \ln p_1(x;\theta)$$

$\Rightarrow \hat{\theta}_{me}$ IS EFFICIENT.

UNDER H_1: $x(t) = n(t) + \theta s(t)$

$$\frac{\delta}{\delta\theta} \ln \Lambda[x(t),\theta] = \int_a^b n(t)q(t)dt$$

$$E_1\left[\left(\frac{\delta}{\delta\theta} \ln \Lambda[x(t),\theta]\right)^2\right]$$

$$= \int_a^b\!\!\int R(t_1,t_2)\,q(t_1)\,q(t_2)\,dt_1\,dt_2$$

$$= \int_a^b s(t)\,q(t)\,dt = d^2 = SNR$$

ERGO, $\mathrm{VAR}\{\hat{\Theta}_{me}\} = \dfrac{1}{\int_a^b s(t)q(t)dt}$

$= 1/d^2$

$\therefore \quad = 1/SNR$

NOTE: THE LARGER THE SNR, THE "BETTER" THE ESTIMATE.

ALSO, ASSUMING H_1'S TRUE:

$$\mathrm{Var}\{\hat{\Theta}_{me}\} = E_1\left\{\left[\frac{\int_a^b (\Theta s(t)+n(t))q(t)dt - \Theta\int_a^b s(t)q(t)dt}{\int_a^b s(t)q(t)dt}\right]^2\right\}$$

$$= E_1\left\{\left[\frac{\int_a^b n(t)q(t)dt}{\int_a^b s(t)q(t)dt}\right]^2\right\}$$

$$= \frac{\int_a^b s(t)q(t)dt}{\left[\int_a^b s(t)q(t)dt\right]^2}$$

$$= \frac{1}{d^2} \quad \Leftarrow \text{SAME ANSWER AS BEFORE}$$

USING $\hat{\Theta}_{me}$ IN THE LOG LIKLIHOOD RATIO, WE GET:

$$\ln \Lambda[x(t), \hat{\Theta}_{me}] = \frac{\left[\int_a^b x(t)q(t)dt\right]^2}{2\int_a^b s(t)q(t)dt}$$

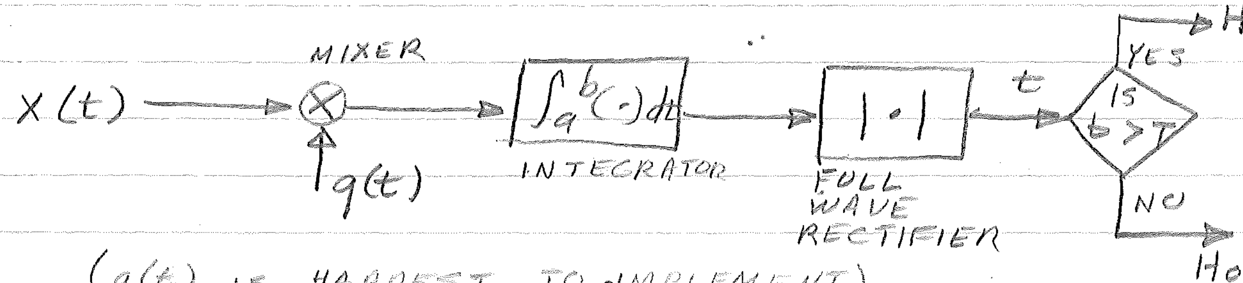
(RECALL: $\ln \Lambda(x(t), \Theta) = \Theta(x,q) - \frac{1}{2}(\Theta^2 s, q)$

$\Rightarrow \hat{\Theta}_{me} = (x,q)/(s,q)$)

NOTE: THE DENOMINATOR IS POSITIVE. THUS, THE TEST IS

$$\left[\int_a^b x(t)q(t)dt\right]^2 \underset{H_0}{\overset{H_1}{\gtrless}} T^2$$

DETECTOR:



(q(t) IS HARDEST TO IMPLEMENT)

READ FERGUSON & VAN TREES.

NO HAIRY PROOFS

12-1-75 (MON)

$$\ln \Lambda[x(t), \theta] = \theta \int_a^b x(t)\,q(t)\,dt - \tfrac{1}{2}\theta^2 \int_a^b s(t)\,q(t)\,dt$$

$$\theta_{me} = \frac{(x,q)}{(s,q)}$$

UNDER H_1

$$\ln \Lambda[x(t), \theta] = \theta \int_a^b [\theta s(t) + n(t)]\,q(t)\,dt - \tfrac{1}{2}\theta^2 \int_a^b s(t)\,q(t)\,dt$$

$$= \tfrac{1}{2}\theta^2 \int_a^b s(t)\,q(t)\,dt + \theta \int_a^b n(t)\,q(t)\,dt$$

$$\frac{\delta}{\delta\theta} \ln \Lambda[x(t), \theta] = \theta \int_a^b s(t)\,q(t)\,dt + \int_a^b n(t)\,q(t)\,dt$$

$$\frac{\delta}{\delta\theta}\, g[x, \theta] = \lim_{h \to 0} \frac{g(x; \theta + h) - g(x, \theta)}{h}$$

BUT THIS IS WRONG! WE MUST NOT TREAT $x(t)$ AS A FUNCTION OF θ

RIGHT WAY TO DO IT IS:

$$\frac{\delta}{\delta\theta} \ln \Lambda[x(t), \theta] = \int_a^b x(t)\,q(t)\,dt - \theta \int_a^b s(t)\,q(t)\,dt$$

THEN

$$E_1\left[\left\{\frac{\delta}{\delta\theta} \ln \Lambda\right\}\right] = d^2$$

$$x(t) \sim N[\theta s(t), \sigma^2]$$

BRIEF REVIEW

- RANDOM PROCESS
- GAUSSIAN RANDOM PROCESS
- MERCER'S THEOREM
- KARHUNEN - LOEVE EXPANSION
- CONTINUOUS TIME DETECTION OF SURE SIGNAL IN GAUSSIAN NOISE

$$H_0: Y(t) = n(t) \quad , H_1: Y(t) = s(t) + n(t)$$

$$\text{TEST STATISTIC} = G = \int_a^b g(t)Y(t)\,dt$$

$$\begin{cases} \text{UNDER } H_0, & G \sim N(0, d^2) \\ \text{UNDER } H_1, & G \sim N(d^2, d^2) \end{cases}$$

$$d^2 = \int_a^b s(t) g(t)\,dt, \qquad s(t) = \int_a^b R(s,t) g(s)\,ds$$

- COMPOSITE HYPOTHESES
- UMP TESTS
- MONOTONE LIKELIHOOD RATIO
- ONE PARAMETER EXPONENTIAL FAMILIES
- CRAMER-RAO LOWER BOUND
- MAXIMUM LIKELIHOOD ESTIMATE
- MAXIMUM LIKELIHOOD DETECTION

TEST WILL PROBABLY HIT TWO ERRORS.

12-3-75.

TEST #2

12-5-75 (FRI)

THIRD PROBLEM

$$\int_a^b s^2(t)\,dt = \sum s_k^2 \le \mathcal{E}$$

$$d^2 = \int_a^b s(t)\,g(t)\,dt = \sum \frac{s_k^2}{\lambda_k}$$

$\sum s_k^2/\lambda_k$ IS MAXIMUM SUBJECT TO $\sum_{k=1}^{\infty} s_k^2 \le \mathcal{E}$

$$s(t) = \sum_{k=1}^{\infty} s_k \phi_k(t)$$

CHOOSE SMALLEST λ_k. LET CORRESPONDING $s_k = \sqrt{\mathcal{E}}$.

ANOTHER WAY:

$$n(t) = \sum_{k=1}^{\infty} n_k \phi_k(t) \qquad Var(n_k) = \lambda_k$$

$$s(t) = \sum_{k=1}^{\infty} s_k \phi_k(t)$$

GRADES: 85, 85, 71, 70, 59, 59, 47

(A WALK NEXT FRIDAY)

NON-PARAMETRIC DETECTION
(DISTRIBUTION FREE, FIXED FALSE ALARM RATE)

GOOD REFERENCES:
1. THOMAS "NONPARAMETRIC DETECTION" PROC. OF IEEE, MAY 1970
2. CARLYLE "NONPARAMETRIC METHODS IN DETECTION IN COMMUNICATION THEORY ED. BY A.V. BALAKRISHNAN

INPUT: $X_1, X_2, \ldots, X_K$

DETECTOR: $\phi(X_1, X_2, \ldots X_K) = \begin{cases} 0 : H_0 \text{ ACCEP} \\ 1 : H_1 \text{ ACCEP} \end{cases}$

POWER FUNCTION $= \beta = E[\phi]$

e.g. H_0: X_K iid $\sim N(0, \sigma^2)$

H_1 X_K iid $\sim N(S_K, \sigma^2)$

N-P. OPTIMUM DETECTOR

$$\left.\begin{aligned} D &= \sum_{K=1}^{K} S_K X_K \\ D &\underset{H_0}{\overset{H_1}{\gtrless}} \end{aligned}\right\} \text{CORRELATION DETECTOR}$$

e.g. H_0: X_K iid $\sim N(0, \sigma^2)$

H_1: X_K iid $\sim N(S, \sigma^2)$; $S > 0$

N-P. OPTIMUM DETECTOR IS

$$D = \sum_{K=1}^{K} X_K$$

DEFINITION: A COMPOSITE HYPOTHESIS IS NON-PARAMETRIC IF IT CAN'T BE SPECIFIED BY A FINITE NUMBER OF PARAMETERS.

DEFINITION: A DETECTOR D IS NONPARAMETRIC WITH RESPECT TO A NON-PARAMETRIC HYPOTHESIS H_0 IF THE FALSE ALARM RATE α IS CONSTANT $\forall\, p_0 \in H_0$

ASSUME X_K ARE iid.

LET $p = P_r[X_K > 0] = 1 - F(0)$

$\ni$ $F(U)$ = DISTRIBUTION FUNCTION

$$= P_r[X \leq U) = \int_{\infty}^{U} f(y)\,dy$$

NONPARAMETRIC HYPOTHESIS

$H_0: p = \frac{1}{2}$, F IS ARBITRARY. (0 MEDIAN)

$H_1: p > \frac{1}{2}$, F IS ARBITRARY

NOTE: THIS IS TESTING ZERO MEDIAN VS. POSITIVE MEDIUM.

LET $f(\cdot)$ BE THE DENSITY OF ANY X_i UNDER H_1.

LET $f^+(\cdot)$ BE THE DENSITY OF ANY X_i UNDER H_1 GIVEN $X_i > 0$

LET $f^-(\cdot)$ BE THE DENSITY OF ANY X_i UNDER H_1 GIVEN $X_i \leq 0$

THEN: $f(x) = p f^+(x) + (1-p) f^-(x)$ UNDER H_1

LET $f_0(x) = \frac{1}{2}\left[f^+(x) + f^-(x)\right]$

NOTE: $f_0(x)$ IS ACCEPTABLE UNDER H_0

CONSIDER THE N-P. TEST OF THE SIMPLE HYPOTHESIS f VS. THE SIMPLE HYPOTHESIS f_0.

$$\Lambda = \prod_{K=1}^{K} \frac{f(X_K)}{f_0(X_K)}$$

NOTE: IF $X_K > 0$, THEN

$$\frac{f(X_K)}{f_0(X_K)} = \frac{p\, f^+(X_K)}{\frac{1}{2} f^+(X_K)} = 2p$$

IF $X_K < 0$, THEN

$$\frac{f(X_K)}{f_0(X_K)} = \frac{(1-p) f^-(X_K)}{\frac{1}{2} f^-(X_K)} = 2(1-p)$$

$$\Rightarrow \Lambda(\vec{x}) = 2^K \cdot p^{\sum_{K=1}^{K} U(X_K)} (1-p)^{K - \sum_{K=1}^{K} U(X_K)}$$

$$\ni U(x) = \begin{cases} 1 & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

$$\Lambda(\vec{x}) = 2^K (1-p)^K \left(\frac{p}{1-p}\right)^{\sum_{K=1}^{K} U(X_K)}$$

THUS THE TEST REDUCES TO

$$\sum_{K=1}^{K} U(X_K) \underset{f_0(\cdot)}{\overset{f(\cdot)}{\gtrless}} T$$

12-8-75 (MON)

WITH "BABY" HYPOTHESIS

$$\sum_{k=1}^{K} U(x_k) \gtrless T$$

$$U(x) = \begin{cases} 0 & , x \le 0 \\ 1 & , x > 0 \end{cases}$$

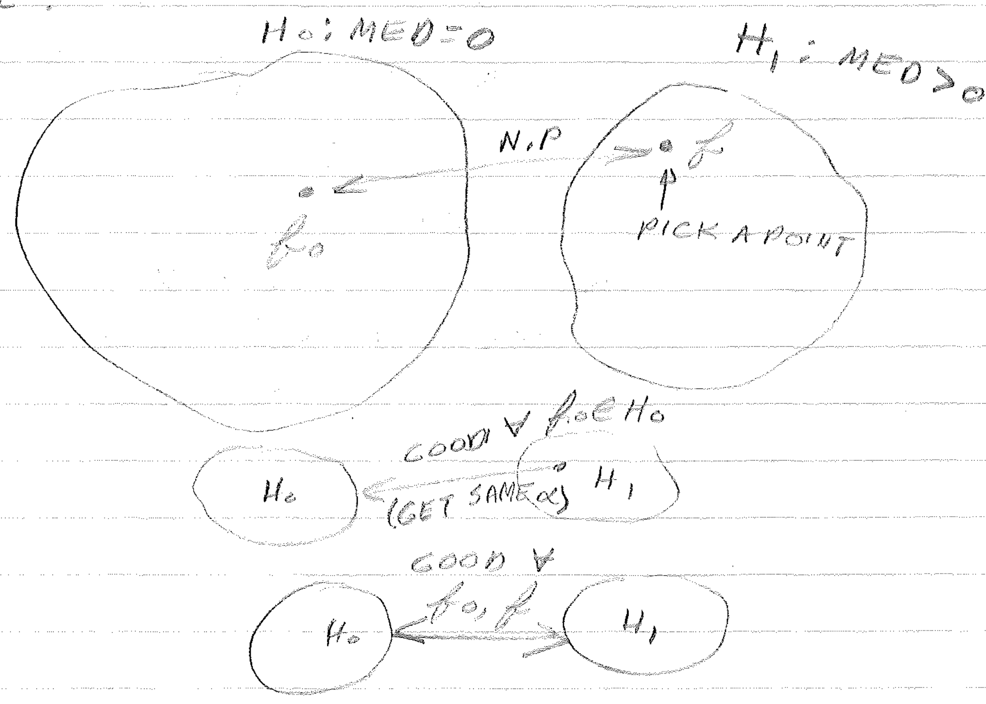
MERELY LOOKS @ SIGN.

FOR SIMPLE HYPOTHESIS f VS. HYP f_0, THIS TEST MAXIMIZES β FOR A FIXED α.

THE SIZE OF THIS TEST WILL BE α $\forall$ H_0.

$\Rightarrow$ THE TEST IS ACTUALLY OPTIMUM FOR THE NON-PARAMETRIC HYPOTHESIS H_0 VS. THE SIMPLE HYPOTHESIS f. HOWEVER, THIS CONCLUSION HOLDS FOR ALL f BELONGING TO H_1.

PICTURE:



SO IN THE CLASS OF HYPOTHESES

H_0 : MEDIAN = 0 (f_0)

H_1 : MEDIAN > 0 (f)

THE SIGN DETECTOR IS NOT ONLY NON-PARAMETRIC, BUT IT IS ALSO OPTIMAL IN THE NEYMAN-PEARSON SENSE.

$$\sum_{k=1}^{K} \mu(x_k) \underset{H_0}{\overset{H_1}{\gtrless}} T \Leftarrow \text{SIGN DETECTOR}$$

TEST STATISTIC:

$$D = \sum_{k=1}^{K} \mu(x_k)$$

D IS DISTRIBUTED BINOMIALLY

UNDER H_0:

$$P_0[D=d] = \binom{K}{d}\left(\frac{1}{2}\right)^d\left(1-\frac{1}{2}\right)^{K-d}$$
$$= \binom{K}{d}\left(\frac{1}{2}\right)^K$$

UNDER H_1:

$$P_1[D=d] = \binom{K}{d} p^d (1-p)^{K-d})$$

$$\alpha = \sum_{d=T+1}^{K} \binom{K}{d}\left(\frac{1}{2}\right)^K \quad \text{(MAY RANDOMIZE)}$$

$$\beta = \sum_{d=T+1}^{K} \binom{K}{d} p^d (1-p)^{K-d}$$

LET g BE A FUNCTION THAT PASSES THROUGH THE ORIGIN AND IS OTHERWISE IS CONFINED TO THE FIRST AND THIRD QUADRANTS. ASSUME THAT THE DETECTOR INPUT IS PERTURBED BY $g(\cdot)$, ie $\tilde{x}_k = g(x_k)$. THEN THE FALSE ALARM PROBABILITY AND DETECTION PROBABILITY ARE UNAFFECTED BY $g(\cdot)$

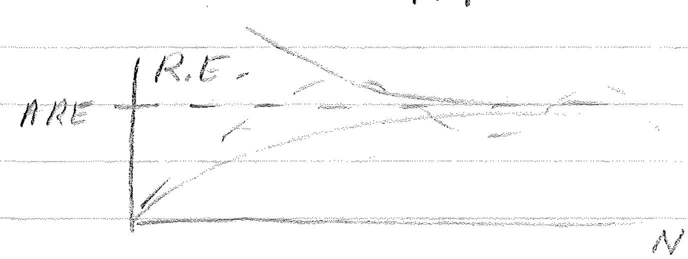
THIS ALSO HOLDS IF THE NON-LINEARITY VARIES WITH TIME (ie $\tilde{x}_K = g(x_K, t)$)

$\Rightarrow$ THIS LEADS TO POPULARITY OF SYSTEMS BASED ON HARD LIMITING.

CONSIDER TWO DETECTORS D_1 AND D_2. LET $N_i(\alpha, \beta)$ DENOTE THE NUMBER OF SAMPLES THAT THE i^{TH} DETECTOR NEEDS TO ACHIEVE A FALSE ALARM PROBABILITY α AND A DETECTION PROBABILITY OF β.

THE ASYMPTOTIC RELATIVE EFFICIENCY OF A DETECTOR D_2 WITH RESPECT TO A REFERENCE DETECTOR D_1 IS

$$ARE_{2,1} = \lim_{\substack{N_1 \to \infty \\ N_2 \to \infty \\ H_1 \to H_0}} \frac{N_1(\alpha, \beta)}{N_2(\alpha, \beta)}$$



LET D_2 BE THE SIGN DETECTOR. APPEAL TO THE CENTRAL LIMIT THEOREM AS K GETS LARGE. UNDER H_0, D_2 HAS THE FOLLOWING DISTRIBUTION

$$D_2 \sim \binom{K}{d}\left(\tfrac{1}{2}\right)^K \quad ; \text{MEAN} = \frac{K}{2} \quad ; \text{VAR} = K/4$$

UNDER H_1

$$D_2 \sim \binom{K}{d} p^d (1-p)^{K-d} \quad ; \text{MEAN} = Kp \quad ; \text{VAR} = Kp(1-p)$$

THEN

$$\alpha_2 = \int_T^\infty \frac{1}{\sqrt{2\pi}\sqrt{K/4}} \exp\left[\frac{-(x-\frac{K}{2})^2}{2 \cdot K/4}\right] dx$$

$$= 1 - \Phi\left(\frac{2T-K}{\sqrt{K}}\right)$$

$$= \Phi\left(\frac{K-2T}{\sqrt{K}}\right)$$

$$\beta_2 = \int_T^\infty \frac{1}{\sqrt{2\pi}\sqrt{Kp(1-p)}} \exp\left[\frac{-(x-Kp)^2}{2Kp(1-p)}\right] dx$$

$$= 1 - \Phi\left[\frac{T-Kp}{\sqrt{Kp(1-p)}}\right]$$

$$= \Phi\left[\frac{Kp-T}{\sqrt{Kp(1-p)}}\right]$$

NOW SUBSTITUTE OUT T:

$$T = \tfrac{1}{2}\left[\sqrt{K}\,\Phi^{-1}(1-\alpha_2) + K\right]$$

$$\Rightarrow \beta_2 = \Phi\left[\frac{Kp}{\sqrt{Kp(1-p)}} - \frac{K+\sqrt{K}\,\Phi^{-1}(1-\alpha_2)}{2\sqrt{Kp(1-p)}}\right]$$

$$= \Phi\left[\frac{\sqrt{K}(2p-1) - \Phi^{-1}(1-\alpha_2)}{2\sqrt{p(1-p)}}\right]$$

12-10-75 (WED)

NOW CONSIDER THE LINEAR DETECTOR D_1:

$$\sum_{K=1}^{\mathbb{K}} X_K \underset{H_0}{\overset{H_1}{\gtrless}} T_1 \quad \Leftarrow \text{0 MEAN WHITE G NOISE}$$

APPEAL TO CENTRAL LIMIT THEOREM AS $\mathbb{K}$ GETS LARGE. ASSUME THAT THE PROBLEM IS CONSTANT ADDITIVE SIGNAL IN ZERO MEAN NOISE.

$$\mathrm{var}(X_K) = \sigma^2$$

UNDER H_0; $E[X_K] = 0$

UNDER H_1; $E[X_K] = \mu > 0$

$$\alpha_1 = \int_{T_1}^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

$$= 1 - \Phi(T/\sqrt{K}\,\sigma)$$

$$\beta_1 = \int_{T_1}^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma} \exp\left(-\frac{(x-K\mu)^2}{2\sigma^2}\right) dx$$

$$= 1 - \Phi\left(\frac{T_1 - K\mu}{\sqrt{K}\,\sigma}\right)$$

$$= \Phi\left(\frac{K\mu - T_1}{\sqrt{K}\,\sigma}\right)$$

SUBSTITUTE OUT T_1:

$$\beta_1 = \Phi\left[\frac{K\mu}{\sqrt{K}\,\sigma} - \frac{\sqrt{K}\,\sigma\,\Phi^{-1}(1-\alpha)}{\sqrt{K}\,\sigma}\right]$$

$$= \Phi\left[\frac{\mu}{\sigma}\sqrt{K} - \Phi^{-1}(1-\alpha_1)\right]$$

$K_1 \to$ LINEAR DET. $\quad K_2 =$ SIGN DETECTOR

SET $\alpha_1 = \alpha_2$ AND $\beta_1 = \beta_2$

FOR $\beta_1 = \beta_2$:

$$\frac{\mu\sqrt{K_1}}{\sigma} - \Phi^{-1}(1-\alpha_1) = \frac{\sqrt{K_2}(2p-1) - \Phi^{-1}(1-\alpha_2)}{2\sqrt{p(1-p)}}$$

$$\frac{\mu}{\sigma}\sqrt{\frac{K_1}{K_2}} = \frac{\Phi^{-1}(1-\alpha_1)}{\sqrt{K_2}} + \frac{(2p-1)}{2\sqrt{p(1-p)}} - \frac{\Phi^{-1}(1-\alpha_2)}{2\sqrt{K_2 p(1-p)}}$$

AS K_1 & $K_2 \to \infty$

$$\sqrt{K_1/K_2} \longrightarrow \frac{(p-\frac{1}{2})}{\sqrt{p(1-p)}}\frac{\sigma}{\mu}$$

$$K_1/K_2 \longrightarrow \frac{(p-\frac{1}{2})^2\sigma^2}{\mu^2 p(1-p)}$$

$$ARE_{2,1} = \lim_{H_1 \to H_0} \frac{(p-\frac{1}{2})^2\sigma^2}{\mu^2 p(1-p)}$$

LET'S ASSUME DENSITIES ARE SYMMETRIC

UNDER H_1:

$$p = P[X_k \leq 0] = \int_{-\infty}^{0} f(x)\,dx$$

$$= \int_{-\infty}^{\mu} f(x-\mu)\,dx$$

$$= \frac{1}{2} + \int_0^{\mu} f(x-\mu)\,dx$$

AS $H_1 \to H_0$, $\mu \to 0$. $\therefore$ $p \to \frac{1}{2} + \mu f(0)$

$$\underset{(H_1 \to H_0)}{ARE_{2,1}} = \frac{[\mu f(0)]^2 \sigma^2}{\mu^2 [\frac{1}{2} + \mu f(0)][\frac{1}{2} - \mu f(0)]}$$

$$= \frac{f(0)^2 \sigma^2}{\frac{1}{4} - \mu^2 [f(0)]^2}$$

$$= 4 f(0)^2 \sigma^2$$

$$\therefore \underline{ARE_{2,1}} = 4 \sigma^2 f(0)^2$$

THIS IS FOR POSITIVE CONSTANT SIGNAL IN NOISE WITH A SYMMETRIC DENSITY FUNCTION $f(\cdot)$

D_2 = SIGN DETECTOR

D_1 = LINEAR DETECTOR

EXAMPLE: TAKE NOISE TO BE GAUSSIAN:

$$f(0) = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$ARE_{2,1} = \frac{2}{\pi} \sim 0.64 \Leftarrow \text{LOWER BOUND}$$

RECALL D_1 IS N.P. OPTIMAL

EXAMPLE: $f(x) = \frac{\gamma}{2} e^{-\gamma |x|}$

$$\sigma^2 = 2/\gamma^2$$

$$f(0)^2 = \gamma^2/4$$

$$ARE_{2,1} = 2$$

SIGN DETECTOR IS TWICE AS GOOD AS LINEAR DETECTOR (IN THE LIMIT)

NOTE:

$$ARE_{2,1} > 1 \quad \text{IF} \quad 4\sigma^2 f(0)^2 > 1$$

NOTE: HEAVY TAILS

A LOT OF ONE'S INTUITION IS BASED ON THE GAUSSIAN ASSUMPTION. DON'T RELY TOO HEAVILY ON INTUITION. D_2 MAY BE DESIRABLE IN AN UNCERTAIN ENVIORNMENT COMPARED TO D_1.

PLUG SHEET FOR TEST I:

$Q_0, Q_1, \alpha, \beta, \pi_0, \pi_1, C_0, C_1, C_{ij}, \Lambda(Y)$

BAYES: $\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\pi_0}{\pi_1}$ $\qquad P_E = \pi_0 Q_0 + \pi_1 Q_1$

WEIGHED COST: $\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\pi_0 C_0}{\pi_1 C_1}$ $\qquad J = C_0 Q_0 \pi_0 + \pi_1 Q_1 C_1$

GENERAL BAYES: $\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})}$

$$R = \pi_0 [C_{00} Q_0 + C_{10}(1-Q_0)] + \pi_1 [C_{11} Q_1 + C_{01}(1-Q_1)]$$

MINIMAX: $C_{01}(1-\hat{Q}_1) + C_{11}\hat{Q}_1 = C_{10}(1-\hat{Q}_0) + C_{00}\hat{Q}_0$

$$\hat{R} = C_{00}\hat{Q}_0 + C_{10}(1-\hat{Q}_0)$$

NEYMAN PEARSON; $\Lambda(Y) \gtrless_{H_0}^{H_1} T$

LOCALLY OPTIMAL: $\sum_{i=1}^{k} -\frac{d}{dx} \ln f(x) \gtrless_{H_0}^{H_1} T$

ADD. WHITE NOISE: $\sum_{i=1}^{k} \ln \frac{f_i(Y_i - S_i)}{f_i(Y_i)} \gtrless_{H_0}^{H_1} T$

LAPLACE NONLINEARITY 

K OBSERVATIONS, WSGN $\Rightarrow$ $f(x) = \frac{1}{\sqrt{2\pi k}\,\sigma} e^{-\frac{(x-k\mu)^2}{2k\sigma^2}}$

TEST $\sum S_i Y_i$ AGAIST $T = \sqrt{k}\,\sigma\,\Phi^{-1}(1-\alpha)$

PLUG SHEET FOR TEST I:

$Q_0, Q_1, \alpha, \beta, \pi_0, \pi_1, C_0, C_1, C_{ij}, \Lambda(Y)$

BAYES: $\Lambda(Y) \gtrless^{H_1}_{H_0} \frac{\pi_0}{\pi_1}$ $\quad P_E = \pi_0 Q_0 + \pi_1 Q_1$

WEIGHED COST: $\Lambda(Y) \gtrless^{H_1}_{H_0} \frac{\pi_0 C_0}{\pi_1 C_1}$ $\quad J = C_0 Q_0 \pi_0 + \pi_1 Q_1 C_1$

GENERAL BAYES: $\Lambda(Y) \gtrless^{H_1}_{H_0} \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$

$$R = \pi_0 \left[C_{00} Q_0 + C_{10}(1 - Q_0) \right] + \pi_1 \left[C_{11} Q_1 + C_{01}(1 - Q_1) \right]$$

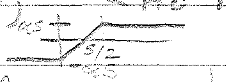
MINIMAX: $C_{01}(1 - \hat{Q}_1) + C_{11}\hat{Q}_1 = C_{10}(1 - \hat{Q}_0) + C_{00}\hat{Q}_0$

$$\hat{R} = C_{00}\hat{Q}_0 + C_{10}(1 - \hat{Q}_0)$$

NEYMAN PEARSON: $\Lambda(Y) \gtrless^{H_1}_{H_0} T$

LOCALLY OPTIMAL: $\sum_{i=1}^{k} -\frac{d}{dx} \ln f(x) \gtrless^{H_1}_{H_0} T$

ADD. WHITE NOISE: $\sum_{i=1}^{k} \ln \frac{f_i(Y_i - S_i)}{f_i(Y_i)} \gtrless^{H_1}_{H_0} T$

LAPLACE NONLINEARITY 

K OBSERVATIONS, WSGN $\rightarrow$ $f(x) = \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{(x - K\mu)^2}{2K\sigma^2}}$

TEST $\sum s_i Y_i$ AGAIST $T = \sqrt{K}\,\sigma\, \Phi^{-1}(1 - \alpha)$

EXPANDED PLUG SHEET FOR TEST I:

$Q_0 = \alpha = \int_T^{\infty} P_0(x)\,dx =$ ERROR OF FIRST KIND / SIZE OF THE TEST / FALSE ALARM PROBABILITY

$Q_1 = 1 - \beta = \int_{-\infty}^{T} P_1(x)\,dx =$ ERROR OF THE SECOND KIND

$\beta = \int_T^{\infty} P_1(x)\,dx =$ DETECTION PROBABILITY / POWER OF THE TEST

H_0 = NULL HYPOTHESIS ; H_1 = ALTERNATE HYPOTHESIS

$\Pi_0 = P[H_0 \text{ OCCURING}]$; $\Pi_1 = P[H_1 \text{ OCCURING}] = 1 - \Pi_0$

$\Lambda(Y) = \dfrac{P_1(Y)}{P_0(Y)}$ = LIKLIHOOD RATIO

- BAYES TEST

$P_e = Q_0\Pi_0 + Q_1\Pi_1$ = TOTAL PROBABILITY OF ERROR

$= \Pi_0 + \int_{R_0} [\Pi_1 P_1(Y) - \Pi_0 P_0(Y)]\,dY$

TO MINIMIZE P_e, MAKE INTEGRAL SMALL AS POSSIBLE:

$$\Lambda(Y) \gtrless_{H_0}^{H_1} \Pi_0/\Pi_1 = T$$

- WEIGHTED COST CRITERION

C_0 = COST OF ANNOUNCING H_1 WHEN H_0 IS TRUE > 0

C_1 = COST OF ANNOUNCING H_0 WHEN H_1 IS TRUE > 0

$J = Q_0 C_0 \Pi_0 + Q_1 C_1 \Pi_1$ = EXPECTED COST

$$\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\Pi_0 C_0}{\Pi_1 C_1} \Rightarrow \text{MINIMIZES } J$$

- GENERAL BAYES CRITERION

C_{ij} = COST OF ANNOUNCING H_i WHEN H_j IS TRUE

FROM WEIGHTED COST CRITERION: $C_0 = C_{10}$, $C_1 = C_{01}$

$\mathcal{R} = \Pi_0 [C_{00} Q_0 + C_{10}(1 - Q_0)] + \Pi_1 [C_{01}(1 - Q_1) + C_{11} Q_1]$

$$\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{(C_{10} - C_{00})\Pi_0}{(C_{01} - C_{11})\Pi_1} \Rightarrow \text{MINIMIZES } \mathcal{R}$$

- MINIMAX CRITERION

$\mathcal{R}(\Pi_1) = \Pi_0 [C_{00} Q_0 + C_{10}(1 - Q_0)] + \Pi_1 [C_{01}(1 - Q_1) + C_{11} Q_1]$

$\mathcal{R}(\hat{\Pi}, \Pi_1) = \Pi_0 [C_{00} \hat{Q}_0 + C_{10}(1 - \hat{Q}_0)] + \Pi_1 [C_{01}(1 - \hat{Q}_1) + C_{11} \hat{Q}_1]$

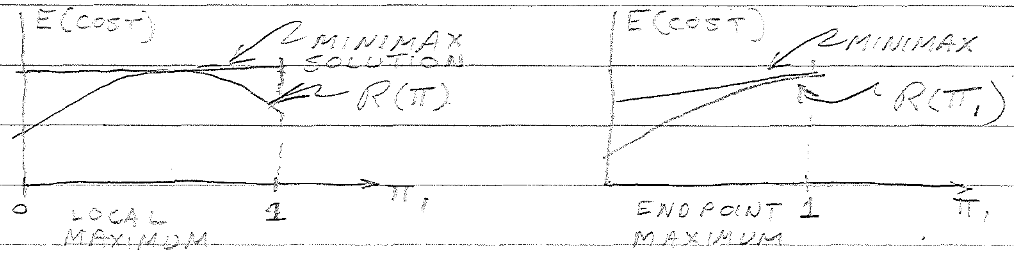
$= [C_{01}(1 - \hat{Q}_1) + C_{11}\hat{Q}_1 - C_{00}\hat{Q}_0 - C_{10}(1 - \hat{Q}_0)]\Pi_1 + C_{00}\hat{Q}_0 + C_{10}(1 - \hat{Q}_0)$

IF $\mathcal{R}(\Pi_1)$ HAS LOCAL MAXIMUM, SET $\mathcal{R}(\Pi_1, \hat{\Pi}) = \text{CONST}$

$\Rightarrow C_{01}(1 - \hat{Q}_1) + C_{11}\hat{Q}_1 = C_{00}\hat{Q}_0 + C_{10}(1 - \hat{Q}_0)$

MINIMAX RISK $= \mathcal{R}(\Pi_1, \hat{\Pi}) = C_{00}\hat{Q}_0 + C_{10}(1 - \hat{Q}_0)$

(CONT)



IF $R(0)$ OR $R(1)$ IS MAXIMUM, USE TANGENT @ 0 OR 1.

$R(\pi_1)$ IS ALWAYS A CONCAVE FUNCTION

- NEYMANN-PEARSON TEST

FOR A GIVEN α, β IS MAXIMIZED BY THE TEST

$$\phi(x) = \text{CRITICAL FUNCTION}$$
$$= P[\text{ANNOUNCING } H_1 \text{ GIVEN OBSERVATION } X]$$
$$= \begin{cases} 1 & ;\ \Lambda(x) > T \\ p & ;\ \Lambda(x) = T \\ 0 & ;\ \Lambda(x) < T \end{cases}$$

$$\alpha = E_0[\phi(x)] \qquad \beta = E_1[\phi_1(x)]$$

$\Lambda(x) \underset{H_0}{\overset{H_1}{\gtrless}} T$ ⇐ NEYMANN PEARSON TEST

- LOCALLY OPTIMAL TEST

USED WHEN SIGNAL IS WEAK COMPARED WITH NOISE

FOR A GIVEN α, $\left.\frac{d\beta}{ds}\right|_{s\to 0} = \frac{d}{ds} E_1[\phi(x)]\Big|_{s\to 0}$ IS MAXIMUM

$\sum_{i=1} -\frac{d}{dx} \ln f(x) \underset{H_0}{\overset{H_1}{\gtrless}} T$ ⇐ WHITE NOISE

$\frac{\partial}{\partial s} P_1(x;s)\big|_{s=0^+} / P_0(x) \underset{H_0}{\overset{H_1}{\gtrless}} T$ ⇐ GENERAL

- NOISE

GAUSSIAN $\sim N(\mu, \sigma^2) \Rightarrow p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

LAPLACE : $p(x) = \frac{\alpha}{2} e^{-|\alpha|/2}$

CAUCHY : $p(x) = \frac{m}{\pi(m^2 + x^2)}$

GENERALIZED LAPLACE NOISE: $\frac{\alpha}{2} e^{-\alpha|x|^k}$

GENERALIZED GAUSSIAN NOISE: $K e^{-K_1|x|^c}$

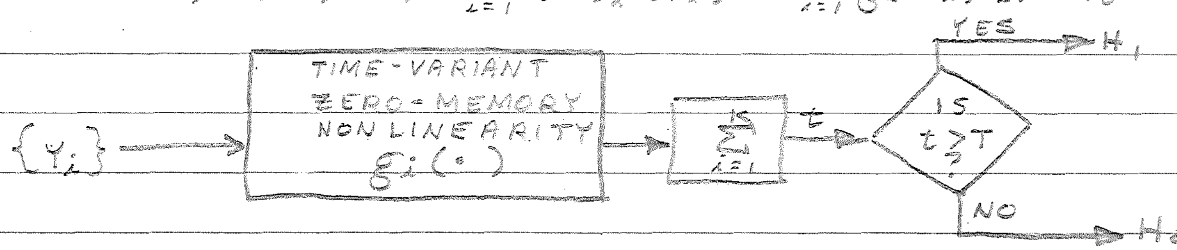
- **DETECTION OF SIGNALS IN ADDITIVE WHITE NOISE**

$H_0 : Y_i = n_i$

$H_1 : Y_i = s_i + n_i \qquad i = 1, 2, \ldots k$

$$\left.\begin{aligned} P_0(Y_1, Y_2, \ldots, Y_K) &= \prod_{i=1}^{k} f_i(Y_i) \\ P_1(Y_1, Y_2, \ldots, Y_K) &= \prod_{i=1}^{k} f_i(Y_i - S_i) \end{aligned}\right\} \text{DUE TO STATISTICAL INDEPENDENCE}$$

$$\ln \Lambda(Y_1, Y_2, \ldots, Y_K) = \sum_{i=1}^{K} \ln \frac{f_i(Y_i - S_i)}{f_i(Y_i)} = \sum_{i=1}^{K} g_i(Y_i, S_i) \underset{H_0}{\overset{H_1}{\gtrless}} T$$



- **DISCRETE TEST**

$H_0 : p_n$

$H_1 : q_n$

$\Lambda_n = \dfrac{q_n}{p_n}$

$$\phi_n = \begin{cases} 1 & \Lambda_n > \Lambda_0 \\ p & \Lambda_n = \Lambda_0 \\ 0 & \Lambda_n < \Lambda_0 \end{cases}$$

$\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \ldots \geq \Lambda_K$

$q_1/p_1 \geq q_2/p_2 \geq \ldots \geq q_K/p_K$

FIND $j \ni \sum_{i=1}^{j} p_i \leq \alpha$

RANDOMIZE $\Rightarrow \sum_{i=1}^{j} p_i + p\,p_{j+1} = \alpha$

- EXAMPLES

1. BAYES CRITERION

ONE OBSERVATION OF CONSTANT SIGNAL

ZERO MEAN GAUSSIAN NOISE : $P(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n^2}{2\sigma^2}}$

$H_0 : Y = n \qquad P_0(Y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{Y^2}{2\sigma^2}}$

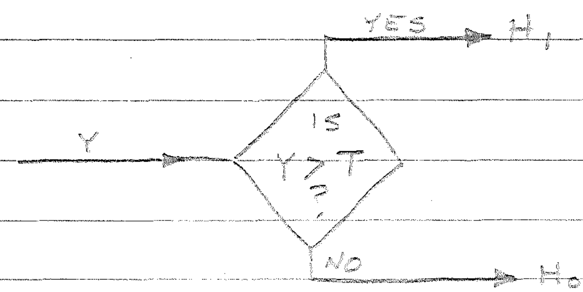
$H_1 : Y = n + S \qquad P_1(Y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(Y-S)^2}{2\sigma^2}}$

$$\Lambda(Y) = \frac{P_1(Y)}{P_0(Y)} = \exp\left[\frac{2SY - S^2}{2\sigma^2}\right]$$

BAYES DETECTOR: $\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\pi_0}{\pi_1}$

$$\exp\left[\frac{2SY - S^2}{2\sigma^2}\right] \gtrless_{H_0}^{H_1} \frac{\pi_0}{\pi_1}$$

$$Y \gtrless_{H_0}^{H_1} \frac{S}{2} + \frac{\sigma^2}{2} \ln \frac{\pi_0}{\pi_1} = T$$



2. WEIGHTED COST CRITERION

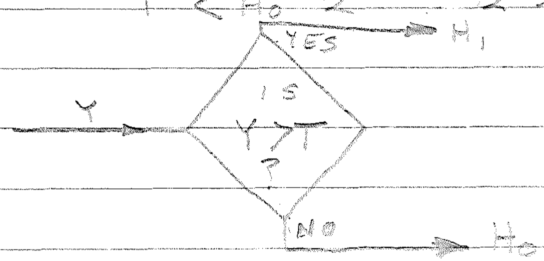
ONE OBSERVATION OF CONSTANT SIGNAL

ZERO MEAN GAUSSIAN NOISE

WEIGHTED COST DETECTOR

$$\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{C_0 \pi_0}{C_1 \pi_1}$$

$$Y \gtrless_{H_0}^{H_1} \frac{S}{2} + \frac{\sigma^2}{2} \ln \frac{\pi_0 C_0}{\pi_1 C_1} = T$$



3. BAYES CRITERION

ONE OBSERVATION OF CONSTANT SIGNAL

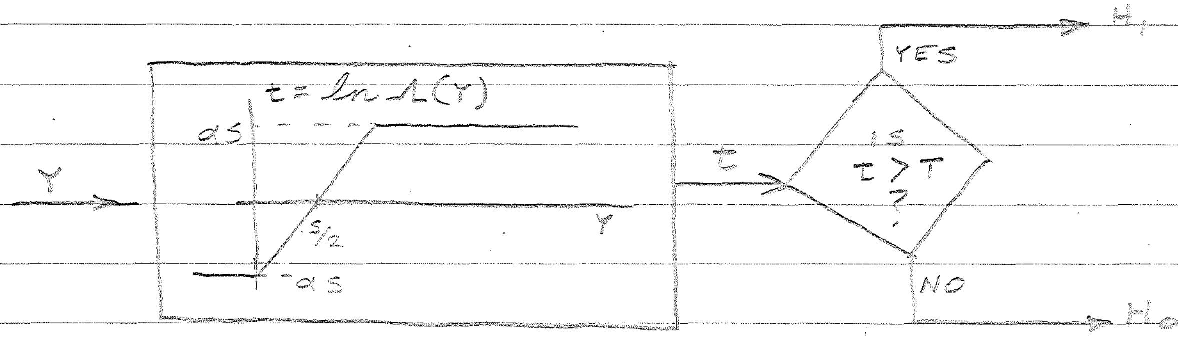
ZERO MEAN LAPLACE NOISE: $P(n) = \frac{\alpha}{2} e^{-\frac{\alpha}{2}|n|}$

$H_0: Y = n$ $\qquad P_0(Y) = \frac{\alpha}{2} e^{-\frac{\alpha}{2}|Y|}$

$H_1: Y = S + n$ $\qquad P_1(Y) = \frac{\alpha}{2} e^{-\frac{\alpha}{2}|Y-S|}$

$$\Lambda(Y) = \left\{ \begin{array}{ll} e^{-\alpha S} & ; Y \le 0 \\ e^{2\alpha Y - \alpha S} & ; 0 \le Y \le S \\ e^{\alpha S} & ; Y \le S \end{array} \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

$$\ln \Lambda(Y) = \left\{ \begin{array}{ll} -\alpha S & ; Y \le 0 \\ \alpha(2Y - S) & ; 0 \le Y \le S \\ \alpha S & ; Y \ge S \end{array} \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{\pi_0}{\pi_1} = T$$



NOTE: IF $T = \ln \frac{\pi_0}{\pi_1} \ge \alpha S$, ALWAYS ANNOUNCE H_1

IF $T = \ln \frac{\pi_0}{\pi_1} \le \alpha S$, ALWAYS ANNOUNCE H_0

4 • BAYES CRITERION

ONE OBSERVATION OF GAUSSIAN R.V. SIGNAL

ZERO MEAN GAUSSIAN NOISE

$$P(n) = \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-Y^2/2\sigma_N^2}$$

$$P(s) = \frac{1}{\sqrt{2\pi}\,\sigma_S} e^{-(Y-m)^2/2\sigma_S^2}$$

$H_0: Y = n \qquad ; \; p_0(Y) = \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-Y^2/2\sigma_N^2}$

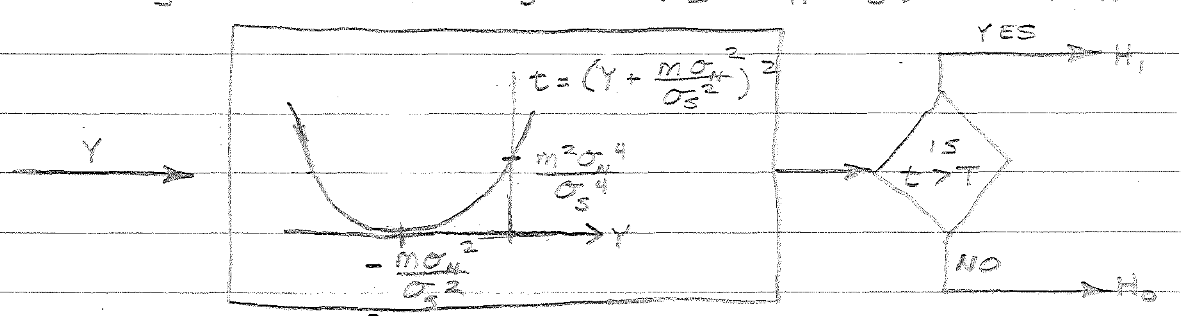
$H_1: Y = s + n \qquad ; \; p_1(Y) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_N^2+\sigma_S^2}} e^{-\frac{(Y-m)^2}{2(\sigma_N^2+\sigma_S^2)}}$

$$\Lambda(Y) = \frac{p_1(Y)}{p_0(Y)} = \frac{\sigma_N}{\sqrt{\sigma_N^2+\sigma_S^2}} \exp\left[\frac{Y^2\sigma_S^2 + 2m\sigma_N^2 Y - m^2\sigma_N^2}{2\sigma_N^2(\sigma_N^2+\sigma_S^2)}\right]$$

$$\Lambda(Y) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_1}{\pi_0}$$

GIVES;

$$\left(Y + \frac{m\sigma_N^2}{\sigma_S^2}\right)^2 \gtrless T = \frac{m\sigma_N^2}{\sigma_S^2} + 2\frac{\sigma_N^2}{\sigma_S^2}(\sigma_N^2+\sigma_S^2)\ln\frac{\pi_0\sqrt{\sigma_N^2+\sigma_S^2}}{\pi_1\sigma_N}$$



5. NEYMAN - PEARSON

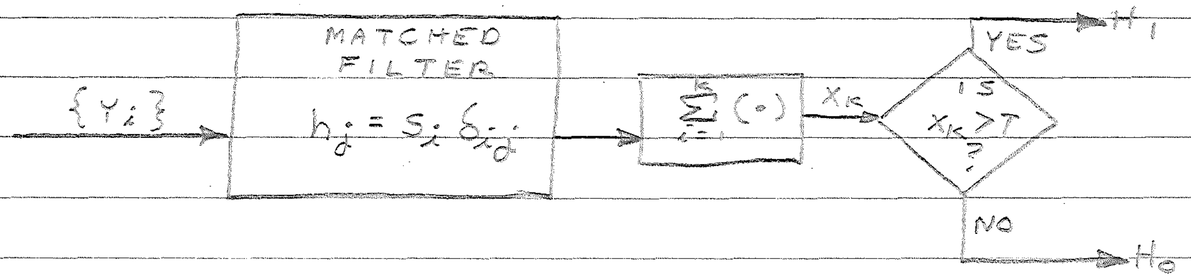
K OBSERVATIONS OF A KNOWN SIGNAL IN ZERO MEAN WHITE STATIONARY GAUSSIAN NOISE

$$H_0 : Y_K = n_K \quad ; \quad P_0(Y_1, Y_2, \ldots Y_K) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^K e^{-\sum_{i=1}^{K} Y_i^2/2\sigma^2}$$

$$H_1 : Y_K = n_K + S_K \quad ; \quad P_1(Y_1, Y_2, \ldots, Y_K) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^K e^{-\sum_{i=1}^{K} \frac{(Y_i - S_i)^2}{2\sigma^2}}$$

$$\Lambda(Y_1, Y_2, \ldots, Y_K) = \exp\, -\frac{1}{2\sigma^2} \sum_{i=1}^{K} \left[(Y_i - S_i)^2 - Y_i^2 \right] \underset{H_0}{\overset{H_1}{\gtrless}} C_0$$

$$\Rightarrow X_K = \sum_{i=1}^{K} Y_i S_i \underset{H_0}{\overset{H_1}{\gtrless}} T$$



6. NEYMAN - PEARSON

K OBSERVATIONS OF A KNOWN POSITIVE CONSTANT SIGNAL IN ZERO MEAN WHITE STATIONARY GAUSSIAN NOISE.

SPECIAL CASE OF ABOVE: $S_i = S > 0$

TEST IS $\sum_{i=1}^{K} Y_i \underset{H_0}{\overset{H_1}{\gtrless}} T$

HERE, $P_0 \sim N(0, K\sigma^2)$; $P_1 \sim N(KS, K\sigma^2)$

$$P_0(Y) = \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{Y^2}{2K\sigma^2}}$$

$$P_1(Y) = \frac{1}{\sqrt{2\pi K}\,\sigma} e^{-\frac{(Y - KS)^2}{2K\sigma^2}}$$

$$\alpha = \int_T^\infty P_0(x)\,dx = 1 - \Phi\left(\frac{T}{\sqrt{K}\,\sigma}\right)$$

WHERE $\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{x^2}{2}}\,dx$

$$\Rightarrow T = \sqrt{K}\,\sigma\, \Phi^{-1}(1 - \alpha)$$

NOTE: WE CAN SET T WITHOUT KNOWING S

$$\beta = \int_T^\infty P_1(x)\,dx = 1 - \Phi\left(\frac{T - KS}{\sqrt{K}\,\sigma}\right)$$

NOTE: WE NEED S, HOWEVER, TO FIND β

7. NEYMAN - PEARSON

TESTING FOR TYPE OF ZERO-MEAN WHITE STATIONARY GAUSSIAN NOISE WITH K OBSERVATIONS.

$H_0;\ \sigma = \sigma_0$

$H_1:\ \sigma = \sigma_1$

$$P_0(x_1, x_2, \dots x_K) = \left(\frac{1}{\sqrt{2\pi}\ \sigma_0}\right)^K e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{K} Y_i^2}$$

$$P_1(x_1, x_2, \dots, x_K) = \left(\frac{1}{\sqrt{2\pi}\ \sigma_1}\right)^K e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{K} Y_i^2}$$

$$\Rightarrow \Lambda(x_1, x_2, \dots, x_K) = \left(\frac{\sigma_0}{\sigma_1}\right)^K \exp\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right) \sum_{i=1}^{K} Y_i^2$$

REDUCES TO

$$\sum_{i=1}^{K} Y_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} R^2$$

NOTE: DECISION BOUNDRY IS A HYPERSPHERE

EQUIVALENTLY: $\sum_{i=1}^{K} \frac{Y_i^2}{\sigma_0^2} \underset{H_0}{\overset{H_1}{\gtrless}} R^2/\sigma_0^2$

$Y_i/\sigma_0 \sim N(0, 1)$

$(Y_i/\sigma_0)^2 \sim \chi_1^2$ = CHI-SQUARED, 1 DEGREE OF FREEDOM

$\sum_{i=1}^{K} (Y_i/\sigma_0)^2 \sim \chi_K^2$ = CHI-SQUARED, K DEGREES OF FREEDOM

$$f_{\chi_K^2}(x) = \frac{x^{\frac{K}{5}-1}\ e^{-s/2}}{2^{K/2}\ \Gamma(K/2)} \quad ; x > 0$$

UNDER NULL HYPOTHESIS

$$\alpha = \int_{\frac{R^2}{\sigma_0^2}}^{\infty} f_{\chi_K^2}(x)\, dx$$

8 ● N.P. : TESTING FOR A WEIGHTED DIE
(RANDOMIZATION EXAMPLE)

H_0: $P_n = \frac{1}{6}$ $n = 1-6$ $\alpha = 0.05$

H_1: $P_1 = 0.2$, $P_n = 0.16$ $n = 2-6$

$$\Lambda(Y) = \begin{cases} 1.2 & ; n = 1 \\ 0.96 & ; n = 2-6 \end{cases}$$

$P_0[\Lambda(Y) = 1.2] = \frac{1}{6}$ $P_1[\Lambda(Y) = 1.2] = 0.2$

$P_0[\Lambda(Y) = 0.96] = \frac{5}{6}$ $P_1[\Lambda(Y) = 0.96] = 0.8$

CRITICAL FUNCTION: $\phi(n) = \begin{cases} 1 & ; \Lambda > \Lambda_0 \\ p & ; \Lambda = \Lambda_0 \\ 0 & ; \Lambda < \Lambda_0 \end{cases}$

$\alpha = E_0[\phi(n)] = 1 \cdot P_0[\Lambda > \Lambda_0] + p \times P_0[\Lambda = \Lambda_0] + 0$

FOR $\alpha \leq \frac{1}{6}$, CHOOSE $\Lambda_0 = 1.2$. @ $\alpha = 0.05$:

$0.05 = 1(0) + \frac{1}{6}p \Rightarrow p = 0.3$

$\beta = E_1[\phi(n)] = 1 \cdot P_1[\Lambda > \Lambda_0] + (0.3) P_1[\Lambda = \Lambda_0] + 0$

$= (0.3)(0.2) = 0.06$

$\Rightarrow$ IF $n = 1$, ANNOUNCE H_1 WITH $\beta = 6\%$

IF $n \neq 2$, ANNOUNCE H_0 WITH $\alpha = 5\%$

9 ● N.P. SAME TEST: TWO OBSERVATIONS n_1, n_2

$$\Lambda(Y) = \begin{cases} 0.04 / \left(\frac{1}{6}\right)^2 = 1.44 & ; n_1 = n_2 = 1 \\ \frac{(0.2)(0.16)}{(1/6)^2} = 1.152 & ; n_1 = 1 \ n_2 \neq 1 \text{ or } n_1 \neq 1, n_2 = 1 \\ \frac{(0.16)^2}{(1/6)^2} = 0.9216 & ; n_1 \neq 1, n_2 \neq 1 \end{cases}$$

$P_0[\Lambda(Y) = 1.44] = \frac{1}{36}$ $P_1[\Lambda(Y) = 1.44] = 0.04$

$P_0[\Lambda(Y) = 1.152] = \frac{10}{36}$ $P_1[\Lambda(Y) = 1.152] = 0.32$

$P_0[\Lambda(Y) = 0.9216] = \frac{25}{36}$, $P_1[\Lambda(Y) = 0.9216] = 0.64$

$\Lambda_0 = 1.152$

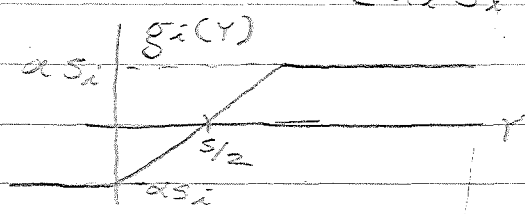
$\alpha = \frac{1}{36} \cdot 1 + p\frac{10}{36} = 0.05 \Rightarrow p = 0.08$

$\beta = 0.04 + (0.08)(0.32) = 0.0656$

10. ADDITIVE WHITE LAPLACE NOISE, K OBSERVATIONS

$$f_i(x) = \frac{\alpha_i}{2} e^{-\alpha_i |x|}$$

$$g_i(x) = \ln \frac{f_i(x-s_i)}{f_i(x)} = \begin{cases} -\alpha_i s_i & ; x \le 0 \\ 2\alpha_i x - \alpha_i s_i & ; 0 < x < s \\ \alpha_i s_i & ; s > 0 \end{cases}$$



11. LOCALLY OPTIMUM.

LAPLACE NOISE: $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$

TEST: $-\frac{d}{dx} \ln \left[\frac{\alpha}{2} e^{-\alpha|x|}\right] = \alpha \, \mathrm{sgn}\, x$

$$\Rightarrow \mathrm{sgn}(x) \underset{H_0}{\overset{H_1}{\gtrless}} T$$

RANDOMIZATION REQUIRED:

$$\phi(x) = \begin{cases} 1 & \mathrm{sgn}(x) > T \\ p & \mathrm{sgn}(x) = T \\ 0 & \mathrm{sgn}(x) < T \end{cases}$$

FOR $0 < \alpha < \frac{1}{2}$, $T = 1$

$\alpha = E_0[\phi(x)] = 0 \cdot 1 + p \cdot \frac{1}{2} + \frac{1}{2} \cdot 0 \Rightarrow \alpha = 2p$

FOR $\frac{1}{2} < \alpha < 1$, $T = -1$

$\alpha = \frac{1}{2}(1) + p \cdot \frac{1}{2} + 0 \cdot 0 \Rightarrow p = 2(\alpha - \frac{1}{2})$

FOR $\alpha = \frac{1}{2}$ $T \in (-1, 1)$

$\alpha = \frac{1}{2}(1) + p'(0) + 0 \cdot \frac{1}{2} = \frac{1}{2}$

12. LOCALLY OPTIMUM

2 OBSERVATIONS IN STATIONARY WHITE LAPLACE NOISE

$$f(x) = c\left[e^{-\alpha(|x_1| + |x_2|)}\right]$$

$$\ln f(x) = \ln c - \alpha\left[|x_1| + |x_2|\right]$$

$$-\frac{d}{dx}\ln f(x) = \alpha\left[\operatorname{sgn} x_1 + \operatorname{sgn} x_2\right]$$

$$t = \operatorname{sgn} x_1 + \operatorname{sgn} x_2 \underset{H_0}{\overset{H_1}{\gtrless}} T$$

$$P_0[t = 2] = 1/4 = P_0[\operatorname{sgn}(x) = 1]\,P_0[\operatorname{sgn}(x) = 1]$$

$$P_0[t = 0] = 1/2 = 2P_0[\operatorname{sgn}(x) = 1]\,P_0[\operatorname{sgn}(x) = -1]$$

$$P_0[t = -2] = 1/4 = P_0[\operatorname{sgn}(x) = -1]\,P_0[\operatorname{sgn}(x) = -1]$$

$$0 < \alpha < \tfrac{1}{4} \Rightarrow T = 2 \Rightarrow \alpha = \tfrac{1}{4}\rho \Rightarrow \rho = 4\alpha$$

$$\tfrac{1}{4} < \alpha < \tfrac{3}{4} \Rightarrow T = 0 \Rightarrow \alpha = \tfrac{1}{4} + \tfrac{1}{2}\rho$$

$$\tfrac{3}{4} < \alpha < 1 \Rightarrow T = -2 \Rightarrow \alpha = \tfrac{3}{4} + \tfrac{1}{4}\rho$$

$\alpha = 1/4 \Rightarrow 0 < T < 2$ (NO RANDOMIZING)

1. HMPP* : A CONSTANT POSITIVE SIGNAL S IS IN WHITE STATIONARY GAUSSIAN NOISE WITH ZERO MEAN. FOR A GIVEN α AND MINIMUM β, HOW MANY OBSERVATIONS, K, ARE NEEDED?

$H_0 : Y_i = n_i$

$H_1 : Y_i = n_i + S$

$$P_0(x) = \frac{1}{\sqrt{2\pi K}\,\sigma_N} e^{-\frac{x^2}{K\sigma_N^2}}$$

$$\alpha = \int_T^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma_N} e^{\frac{-x^2}{2K\sigma_N^2}} dx$$

$$= \int_{\frac{T}{\sqrt{K}\sigma_N}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= 1 - \Phi\left[\frac{T}{\sqrt{K}\,\sigma_N}\right] \ni \Phi(\xi) = \int_{-\infty}^{\xi} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\Rightarrow \frac{T}{\sqrt{K}\,\sigma_N} = \Phi^{-1}[1-\alpha]$$

$$T = \sqrt{K}\,\sigma_N \Phi^{-1}[1-\alpha]$$

$$\beta = \int_{T}^{\infty} \frac{1}{\sqrt{2\pi K}\,\sigma_N} e^{-(x-KS)^2/2K\sigma_N^2} dx$$

$$= \int_{\frac{T-KS}{\sqrt{K}\sigma_N}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= 1 - \Phi\left[\frac{T-KS}{\sqrt{K}\,\sigma_N}\right]$$

$$\frac{T-KS}{\sqrt{K}\,\sigma_N} = \Phi^{-1}[1-\beta]$$

$$T = \sqrt{K}\,\sigma_N \Phi^{-1}[1-\beta] + KS$$

THUS K MUST SATISFY

$$\sqrt{K}\,\sigma_N \Phi^{-1}[1-\alpha] = \sqrt{K}\,\sigma_N \Phi^{-1}[1-\beta] + KS$$

$$K\sigma_N^2\left[\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)\right]^2 = K^2 S^2$$

$$K = \frac{\sigma_N^2}{S^2}\left[\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)\right]^2$$

*HOMEMADE PRACTICE PROBLEM

2. HMPP. TESTING FOR A WEIGHTED COIN

$H_0 : p_h = \frac{1}{2} \;,\; p_t = \frac{1}{2}$

$H_1 : p_h = \frac{1}{4} \;;\; p_t = \frac{3}{4}$

ONE OBSERVATION

$$\Lambda = \begin{cases} \frac{1}{2} & ; h \\ \frac{3}{2} & ; t \end{cases}$$

$$P_0[\Lambda = \tfrac{1}{2}] = \tfrac{1}{2} \qquad P_0[\Lambda = \tfrac{3}{2}] = \tfrac{1}{2}$$

$$\phi = \begin{cases} 1 & , \Lambda > \Lambda_0 \\ p & , \Lambda = \Lambda_0 \\ 0 & , \Lambda < \Lambda_0 \end{cases}$$

$\alpha = E_0[\phi]$

CHOOSE $\Lambda_0 = \frac{3}{2}$ FOR $0 < \alpha < \frac{1}{2}$

$\alpha = 1 \cdot 0 + p(\frac{1}{2}) + 0 \cdot \frac{1}{2} \Rightarrow p = 2\alpha$

CHOOSE $\Lambda_0 = \frac{1}{2}$ FOR $\frac{1}{2} < \alpha < 1$

$\alpha = 1 \cdot \frac{1}{2} + p \cdot \frac{1}{2} + 0 \cdot 0 \Rightarrow p = 2(\alpha - \frac{1}{2})$

FOR β:

$P_1[\Lambda = \frac{1}{2}] = \frac{1}{4} \;,\; P_1[\Lambda = \frac{3}{2}] = \frac{3}{4}$

FOR $0 < \alpha < \frac{1}{2}$, $p = 2\alpha$, $\Lambda_0 = \frac{3}{2}$

$\beta = 1 \cdot 0 + (2\alpha)\frac{3}{4} + 0 = \frac{3}{2}\alpha$

FOR $\frac{1}{2} < \alpha < 1$, $p = 2\alpha - 1$, $\Lambda_0 = \frac{1}{2}$

$\beta = 1 \cdot \frac{3}{4} + (2\alpha - 1)\frac{1}{4} = \frac{\alpha + 1}{2}$

3. HMPP. TESTING FOR A WEIGTED COIN

ONE OBSERVATION, GENERAL CASE

$H_0 : P_{h0} = \frac{1}{2} \quad P_{t0} = \frac{1}{2}$

$H_1 : P_{h1}, \quad P_{t1} = 1 - P_{h1} < P_{h1}$

$$\Lambda = \begin{cases} 2P_{h1} \\ 2P_{t1} \end{cases}$$

$P_0[\Lambda = 2P_{h1}] = \frac{1}{2} \quad ; \quad P_0[\Lambda = 2P_{t1}] = \frac{1}{2}$

$P_1[\Lambda = 2P_{h1}] = P_{h1} \quad ; \quad P_1[\Lambda = 2P_{t1}] = P_{t1}$

$$\phi = \begin{cases} 1 & ; \Lambda > \Lambda_0 \\ P & ; \Lambda = \Lambda_0 \\ 0 & ; \Lambda < \Lambda_0 \end{cases}$$

a. $0 < \alpha < \frac{1}{2} \Rightarrow$ CHOOSE $\Lambda_0 = 2P_{h1}$

$\alpha = 0 \cdot 1 + \frac{1}{2}P + 0 \cdot \frac{1}{2} \Rightarrow P = 2\alpha$

$\beta = 0 \cdot 1 + P_{h1}(2\alpha) + 0 \cdot P_{t1} = 2\alpha P_{h1}$

b. $\frac{1}{2} < \alpha < 1 \Rightarrow$ CHOOSE $\Lambda_0 = 2P_{t1}$

$\alpha = \frac{1}{2} + \frac{1}{2}P + 0 \cdot 0 \Rightarrow P = 2\alpha - 1$

$\beta = 1 \cdot P_{h1} + 2P_{t1}P_{h1}\alpha = P_{h1} + 2(1 - P_{h1})P_{h1}\alpha$

4. HMPP: TESTING FOR A WEIGHTED COIN

TWO OBSERVATIONS

$$H_0: p_n = \frac{1}{2} \quad ; n = 1, 2$$

$$H_1: p_0 = \frac{1}{4}, \quad p_1 = \frac{3}{4}$$

OR $H_0: \quad P_{00} = \frac{1}{4} \quad P_{10} = \frac{1}{2} \quad P_{11} = \frac{1}{4}$

$H_1: \quad P_{00} = \frac{1}{16} \quad P_{10} = \frac{6}{16} \quad P_{11} = \frac{9}{16}$

$$\Lambda = \begin{cases} \frac{1}{4} & n_1 = n_2 = 0 \\ \frac{3}{4} & n_1 = 1 \ n_0 = 0 \text{ OR } n_1 = 0,\ n_0 = 1 \\ \frac{9}{4} & n_1 = n_2 = 1 \end{cases}$$

$P_0(\Lambda = \frac{1}{4}) = \frac{1}{4} \quad P_0(\Lambda = \frac{3}{4}) = \frac{1}{2} \quad P_0(\Lambda = \frac{9}{4}) = \frac{1}{4}$

$P_1(\Lambda = \frac{1}{4}) = \frac{1}{16} \quad P_1(\Lambda = \frac{3}{4}) = \frac{6}{16} \quad P_1(\Lambda = \frac{9}{4}) = \frac{9}{16}$

$$\phi = \begin{cases} 1 & ; \Lambda > \Lambda_0 \\ p & ; \Lambda = \Lambda_0 \\ 0 & ; \Lambda < \Lambda_0 \end{cases}$$

a. $0 < \alpha < \frac{1}{4} \Rightarrow \Lambda_0 = \frac{9}{4}$

$$\alpha = 0 \cdot 1 + \frac{1}{4} p + 0 \cdot \frac{3}{4} \Rightarrow p = 4\alpha$$

$$\beta = 0 \cdot 1 + \frac{9}{16}(4\alpha) + 0 \Rightarrow \beta = \frac{9}{4}\alpha$$

b. $\frac{1}{4} < \alpha < \frac{1}{2} \Rightarrow \Lambda_0 = \frac{3}{4}$

$$\alpha = \frac{1}{4} \cdot 1 + \frac{1}{2} p + 0 \Rightarrow p = \frac{2\alpha - 1}{2}$$

$$\beta = \frac{9}{16} + \frac{6}{16}\left(\frac{2\alpha - 1}{2}\right) \quad \text{ETC.}$$

CONDENSED REVIEW SHEET: TEST II:

- RANDOM PROCESS

$E[|X(t)|^2] < \infty \Leftarrow$ SECOND ORDER RANDOM PROCESS

$R(t,s) = E[X(s), X(t)] \Leftarrow$ AUTOCORRELATION

$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j R(t_i, t_j) \geq 0 \Leftarrow$ POSITIVE DEFINATE

FOLLOWS FROM: $E\left[\left(\sum_{i=1}^{n} a_i X(t_i)\right)^2\right] \geq 0$

$F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P_r[X(t_1) > x_1, \ldots, X(t_n) > x_n]$

- MERCER'S THEOREM

$R(t,s) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\phi_n(s)$

$\lambda_n \phi_n(t) = \int_a^b R(t,s)\phi_n(s)\,ds$

- KAHUNEN - LOEVE EXPANSION

$X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \Leftarrow$ SECOND ORDER PROCESS

$E[X_n, X_m] = \delta_{nm}\lambda_n$

- CONTINUOUS TIME DETECTION OF SURE SIGNAL IN GAUS. NOISE

$s(t) = \int_a^b R(t,s) q(s)\,ds \Leftarrow$ PSEUDO SIGNAL

$H_0: Y(t) = n(t)$; $H_1: Y(t) = n(t) + s(t)$

$G = \int_a^b Y(t) q(t)\,dt = \sum_{n=1}^{\infty} Y_K q_K = \sum_{n=1}^{\infty} \frac{Y_K S_K}{\lambda_K} \Leftarrow$ TEST STATIS.

$E_0[Y_K] = 0, \quad E_1[Y_K] = S_K, \quad var(Y_K) = \lambda_K$

$E_0[G] = 0, \quad E_1[G] = d^2 = \sum_{n=1}^{\infty} \frac{S_K^2}{\lambda_K}; \quad var(G) = d^2$

NOTE: $d^2 = \sum_{n=1}^{\infty} S_K q_K = \int_a^b s(t) q(t)\,dt$

$\ln \Lambda = \sum_{K=1}^{\infty} \frac{1}{\lambda_K}\left[S_K X_K - \frac{1}{2} S_K^2\right]$

$= \int_a^b \left[q(t)x(t) - \frac{1}{2} q(t)s(t)\right] dt$

TEST IS

$$G \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0 + \frac{1}{2}\int_a^b q(t)s(t)\,dt$$

$\Rightarrow$ TWO SIGNAL DETECTION

$H_0: n(t) + s_0(t)$; $H_1: n(t) + s_1(t)$

TEST IS

$$\int_a^b x(t)[q_1(t) - q_0(t)] \gtrless \ln\Lambda_0 + \frac{1}{2}\int_a^b [s_1 q_1 - s_0 q_0]$$

- COMPOSITE HYPOTHESIS

BAYES TEST

$H_0: p(x), \pi_0, C_{10}, C_{00}$ $\quad H_1: p(x;\theta), \pi_1, C_{01}(\theta), C_{11}(\theta)$

$C_{10} > C_{00}$ $\quad C_{01}(\theta) > C_{11}(\theta)$

COST OF
- 00: $\pi_0 C_{00} \int_{R_0} p(x)dx$
- 10: $\pi_0 C_{10} \int_{R_1} p(x)dx$
- 01: $\pi_1 \int\int_{R_0} C_{01}(\theta) z(\theta) p(x;\theta) d\theta dx$
- 11: $\pi_1 \int\int_{R_1} C_{11}(\theta) z(\theta) p(x;\theta) d\theta dx$

MINIMIZING EXPECTED COST

$$\frac{\pi_1 \int [C_{01}(\theta) - C_{11}(\theta)] p(x;\theta) z(\theta) d\theta}{\pi_0 [C_{10} - C_{00}] p(\theta)} \underset{H_1}{\overset{H_0}{\gtrless}} 1$$

- UMP TEST

TEST IS UMP (SIZE α) FOR $H_0: \theta \in \Theta_0$, $H_1: \theta \in \Theta_1$

IF CAN DEFINE N-P TEST w/o θ

$\alpha = \sup_{\theta \in \Theta_0} E_0[\phi_0(x)]$; $\phi_0(x) \equiv$ N-P TEST

$E_1[\phi_0] \geq E_1[\phi(x)] \ \forall \ \theta \in \Theta_1$

- MONOTONE LIKELIHOOD RATIO

IF $\Lambda = \frac{p(x;\theta_1)}{p(x;\theta_0)}$ IS NON-DECREASING $\forall \ \theta_1 > \theta_0$

$H_0: \theta \leq \theta_T \quad H_1: \theta > \theta_T$

THEN $\phi(x) = \begin{cases} 1 ; & x > x_T \\ \rho ; & x = x_T \\ 0 ; & x < x_T \end{cases}$

IS UMP

- ONE PARAMETER EXPONENTIAL FAMILY

  $$p(x;\theta) = h(x)\, C(\theta) \exp Q(\theta)\, T(x)$$

  IF $Q(\theta)$ AND $T(x)$ ARE MONO. INCREASING

  $\Rightarrow p(x;\theta)$ HAS MONOTONE L.R.

- SUFFIENT STATISTIC
  (FACTORIZATION THEOREM)

  $T(x)$ IS SUFFICIENT IF $p(x;\theta) = g[T(x),\theta]\, h(x)$

  IN OPEF, $T(x)$ IS SUFFICIENT

- UNBIASED : $E[\hat{\theta}] = \theta$

  CRAMER-RAO: $\quad var(\hat{\theta}) \geq \dfrac{1}{E[\{\frac{\delta}{\delta\theta} \ln p(x;\theta)\}^2]}$

  EFFICIENT: $\quad \frac{\delta}{\delta\theta} \ln p(x;\theta) = K(\theta)\,[\hat{\theta} - \theta]$

  ($var\,\hat{\theta}$ = CRLB)

- MAXIMUM LIKELIHOOD

  $$\frac{\delta}{\delta\theta} \ln p(x;\theta)\Big|_{\theta=\theta_{me}} = 0$$

  IFF EFFICIENT ESTIMATE EXISTS, IT IS $\hat{\theta}_{me}$

  $T(x)$ IS EFFIENT ESTIMATE IN OPEF

- MAXIMUM LIKELIHOOD DETECTOR

  $$\frac{p[x, \hat{\theta}_{me\,1}]}{p[x, \hat{\theta}_{me\,0}]} \gtrless_{H_1}^{H_0} T$$

- IN GAUS. NOISE

  $H_0:\ x(t) = n(t) \qquad H_1:\ x(t) = n(t) + \theta s(t)$

  $$\hat{\theta}_{me} = \int_a^b x(t) g(t)\,dt \Big/ \int_a^b s(t) g(t)\,dt$$

EXPANDED PLUG SHEET FOR TEST II

- RANDOM PROCESSES

AUTOCORRELATION: $R(t,s) = E[X(t)X(s)]$

$R(t,s)$ IS NON-NEGATIVE DEFINATE

- L_2 THEORY

$f(t) \in L_2 \Rightarrow \|f\|^2 < \infty$

$\|f\| = \text{NORM} = \sqrt{\int_a^b |f(t)|^2 dt} = \sqrt{(f,f)}$

$(f,g) = \text{INNER PRODUCT} = \int_a^b f(t)g(t)dt$

- ORTHOGONALITY

$(f_n, f_m) = 0 \quad \forall \; n \neq m$

- ORTHONORMAL

$\|f_n\| = 1$

- ORTHONORMAL REPRESENTATION

$g(t) = \sum_n a_n f_n(t) \ni a_n = (g(t), f_n(t))$

- CONVERGENCE IN L_2 NORM

$\lim_{N \to \infty} \|g(t) - \sum_{n=1}^{N} a_n f_n(t)\| = 0$

- BESSEL'S INEQUALITY

$\|h(t)\| \geq \sum_{n=1}^{N} (h_n)^2 \ni h_n = (h(t), f_n(t))$

- COMPLETE ORTHONORMAL SET IF

$\|h(t)\| = \sum_{n=1}^{\infty} (h_n)^2$

- PARCEVAL'S THEOREM

IF $\{f_n(t)\}$ IS A COMPLETE ORTHONORMAL SET,

THEN, IF, $g(t), h(t) \in L_2$,

$(g,h) = \sum g_n h_n$

$g_n = (f_n, g) \; ; \; h_n = (f_n, h)$

- MERCER'S THEOREM

IF $R(t,s)$ IS SYMMETRIC, CONTINUOUS, AND NON-NEGATIVE DEFINATE IN $a<(t,s)<b$,

$$R(t,s) = \sum_{n=1} \lambda_n \phi_n(t)\phi_n(s) \qquad ; a<(t,s)<b$$

λ_n = EIGENVALUE

$\phi_n(t)$ = EIGENFUNCTION:

$$\lambda_n \phi_n(s) = \int_a^b R(t,s)\phi_n(t)\,dt$$

- KARHUNEN-LOEVE EXPANSION

$X(t) = 2^{nd}$ ORDER RANDOM PROCESS

$$X(t) = \sum_{n=1}^{\infty} x_n \phi_n(t) \qquad a \le t \le b$$

- CONVERGENCE IN MEAN SQUARE:

$$\lim_{N\to\infty} E\left[\left(X(t) - \sum_1^N x_n\phi_n(t)\right)^2\right] = 0$$

$$x_n = (X(t), \phi_n(t))$$

$$E[x_n, x_m] = \lambda_n \delta_{nm} \Leftarrow x_n\text{'s UNCORRELATED}$$

$$\lambda_n \ge 0, \quad \sum_n \lambda_n^2 < \infty$$

- $\phi_n(t)$ FROM $\lambda_n\phi_n(t) = \int_a^b R(t,s)\phi_n(s)\,ds$

- INTEGRAL EQUATION SOLUTION

$$s(t) = \int_a^b R(t,\tau)\, q(\tau)\,d\tau$$

$$\lambda_n \phi_n(t) = \int_a^b R(t,\tau)\phi_n(\tau)\,d\tau$$

$$s(t) = \sum_n s_n\phi_n(t) \qquad s_n = (\phi_n, s)$$

$$q(t) = \sum_n q_n\phi_n(t) \qquad q_n = (q, \phi_n)$$

THEN: $\sum s_n\phi_n = \sum_n q_n\lambda_n\phi_n$

$$s_n = q_n\lambda_n \Rightarrow q_n = s_n/\lambda_n \; ; \sum_n q_n^2 < \infty$$

$$\Rightarrow q(t) = \sum_n (s_n/\lambda_n)\,\phi_n$$

SPECIFYING $q(t)$ SPECIFIES $s(t)$

CONVERSE IS NOT TRUE

- TEST STATISTIC DERIVATION

$H_0: Y(t) = n(t)$

$H_1: Y(t) = n(t) + Y(t)$

- NOISE IS 1. ZERO MEAN. 2) GAUSSIAN

3) CONTINUOUS POSITIVE DEFINATE $R(t,s)$

- DEFINE

SIGNAL: $s(t) = \sum_k s_k \phi_k(t) \ni s_k = (s, \phi_k)$

PSEUDO SIGNAL: $q(t) = \sum_k q_k \phi_k(t) \ni q_k = (q, \phi_k) = \frac{s_k}{\lambda_k}$

OBSERVATION: $Y(t) = \sum_k Y_k \phi_k(t) \ni Y_k = (Y, \phi_k)$

- APRIORI KNOWLEDGE:

$E_0[Y_k] = 0$ (SINCE NOISE IS ZERO MEAN)

$E_1[Y_k] = s_k$

$cov_0[Y_k, Y_j] = cov_1[Y_k, Y_j] = \lambda_k \delta_{jk} \Leftarrow$ UNCORRELATED

- OBSERVABLES: $Y_1, Y_2, \ldots, Y_K$

$\Rightarrow p_0(\vec{Y}_k) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp \frac{-Y_k^2}{2\lambda_k}$

$p_1(\vec{Y}_k) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi\lambda_k}} \exp \frac{-(Y_k - s_k)^2}{2\lambda_k}$

$\Lambda(\vec{Y}_k) = \exp \sum_{k=1}^{K} \frac{1}{2\lambda_k}\left[2Y_k s_k - s_k^2\right]$

- TEST (NEYMANN-PEARSON) IS

$\Lambda \gtrless_{H_0}^{H_1} \Lambda_0$

PUTTING IN A MORE PALATABLE FORM:

LET $G_K \triangleq \sum_{k=1}^{K} \frac{s_k Y_k}{\lambda_k} \Leftarrow$ TEST STATISTIC

$\Rightarrow G_K \gtrless_{H_0}^{H_1} \ln \Lambda_0 + \frac{1}{2} \sum_{k=1}^{K} \frac{s_k^2}{\lambda_k}$

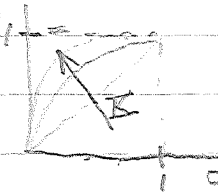
LET d_K^2 = SIGNAL TO NOISE RATIO

WHERE $d_K^2 = \sum_{k=1}^{K} \frac{s_k^2}{\lambda_k}$

THEN TEST BECOMES

$G_K \gtrless_{H_1}^{H_0} \ln \Lambda_0 + \frac{1}{2} d_K^2$

- NOTE: AS K INCREASES, N-P CURVE GETS FATTER $\longrightarrow$



TEST STATISTIC DERIVATION (CONT.)

- THINGS ABOUT TEST STATISTIC, G_K

$$E_1[G_K] = E_0\left[\sum_{k=1}^{K} \frac{Y_k S_k}{\lambda_k}\right] = \sum_{k=1}^{K} \frac{S_k^2}{\lambda} = d_K^2$$

$$E_0[G_K] = 0$$

$$\mathrm{var}_0[G_K] = \mathrm{var}_1[G_K] = d_K^2$$

CONVERGENCE OF G_K IS MEAN SQUARE WITH PROBABILITY ONE.

- IN TIME, THE DETECTOR IS

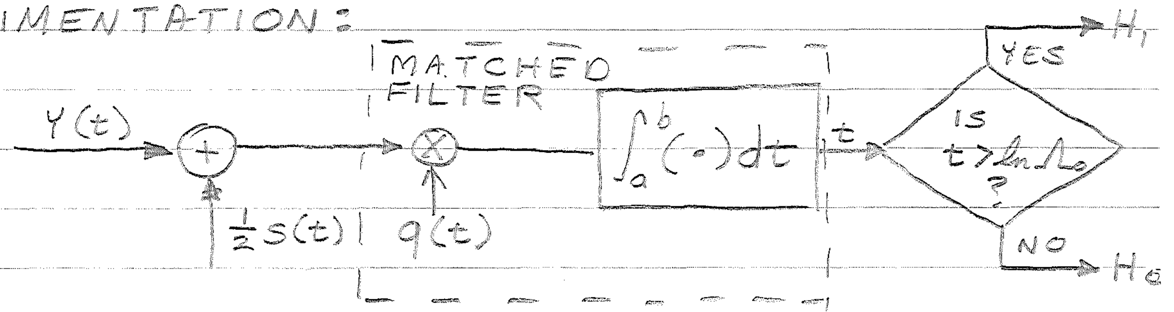
$$G[Y(t)] \gtrless_{H_0}^{H_1} \ln \Lambda_0 + \frac{1}{2}\int_a^b s(t)\,q(t)\,dt$$

WHERE $G[Y(t)] = \int_a^b q(t)\,Y(t)\,dt$

AND $s(t) = \int_a^b R(t,s)\,q(s)\,ds$

EQUIVALENTLY $\int_a^b q(t)\left[Y(t) - \frac{1}{2}s(t)\right] \gtrless_{H_0}^{H_1} \ln \Lambda_0$

IMPLIMENTATION:



- SIGNAL TO NOISE RATIO (63)

$$SNR = \frac{\left[\int_a^b s(t)\,q(t)\right]^2}{E\left[\int_a^b n(t)\,q(t)\,dt\right]^2} = \frac{\text{SIGNAL ENERGY}}{\text{NOISE ENERGY}}$$

$$= \left[\int_a^b s(t)q(t)\,dt\right]^2 \Big/ E\left[\int_a^b\int_a^b n(t_1)\,n(t_2)\,q(t_1)\,q(t_2)\,dt_1\,dt_2\right]$$

$$= \left[\int_a^b s(t)q(t)\,dt\right]^2 \Big/ \int_a^b\int_a^b E[n(t_1)n(t_2)]\,q(t_1)\,q(t_2)\,dt_1\,dt_2$$

$$= \left[\int_a^b s(t)q(t)\,dt\right]^2 \Big/ \int_a^b\int_a^b R(t_1,t_2)\,q(t_1)\,q(t_2)\,dt_1\,dt_2$$

$$= \left[\int_a^b s(t)q(t)\,dt\right]^2 \Big/ \int_a^b s(t_2)\,q(t_2)\,dt_2$$

$$= \int_a^b s(t)\,q(t)\,dt$$

$$= \sum_{k=1}^{\infty} q_k S_k$$

$$= \sum_{k=1}^{\infty} \frac{S_k^2}{\lambda_k} = d^2$$

- DETECTION OF TWO SIGNALS

$$H_0: \; Y(t) = s_0(t) + n(t)$$
$$H_1: \; Y(t) = s_1(t) + n(t)$$

GOING THROUGH STEPS AS BEFORE GIVES

$$G = \int_0^b Y(t)\,[q_1(t) - q_0(t)]\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \ln \Lambda_0 + \frac{1}{2}\int_0^b [s_1(t)q_1(t) - s_0(t)q_0(t)]\,dt$$

WHERE $s_0(t_1) = \int_0^b R(t_1, t_2)\, q_0(t_2)\, dt_2$

$$s_1(t_1) = \int_0^b R(t_1, t_2)\, q_1(t_2)\, dt_2$$

- GAUSSIAN NOISE WITH TIME-VARIANT MEAN

WRITE $n(t) = n_0(t) + m(t)$

ASSSOCIATE $m(t)$ WITH THE SIGNAL

- ZERO-MEAN STATIONARY WHITE GAUSSIAN NOISE

$$R(t_1, t_2) = N_0\, \delta(t_1 - t_2)$$
$$s(t_2) = \int_0^b R(t_1, t_2)\, q(t_2)\, dt_2 = N_0 q(t_1)$$

DETECTOR IS THUS

$$G = \frac{1}{N}\int_0^b Y(t)\, s(t)\, dt \underset{H_0}{\overset{H_1}{\gtrless}} T$$

- BAYES TEST: COMPOSITE HYPOTHESIS

$H_0: \; p_0(x)$. (SIMPLE)

$H_1: \; p_1(x, \theta)$ (COMPOSITE) ; θ = UNKNOWN PARAMETER

- ASSUME WE KNOW: π_0, π_1, P.D.F OF $\theta = z(\theta)$,

  COSTS C_{10}, C_{00}, $C_{01}(\theta)$, $C_{11}(\theta)$

  (C_{ij} = COST OF SAYING i WHEN j IS TRUE)

- ASSUME: $C_{10} > C_{00}$; $C_{01}(\theta) > C_{11}(\theta)$ {ii: CORRECT DECISION COSTLESS

- EXPECTED COSTS ARE THEN

$$\begin{cases} \text{COST OF } H_0/H_0 = \pi_0 C_{00} \int_{R_0} p_0(x)\,dx \\ \text{COST OF } H_1/H_0 = \pi_0 C_{10} \int_{R_1} p_0(x)\,dx \\ \text{COST OF } H_0/H_1 = \pi_1 \int_{R_0}\int p_1(x,\theta)\, C_{01}(\theta)\, z(\theta)\,dx\,d\theta \\ \text{COST OF } H_1/H_1 = \pi_1 \int_{R_1}\int p_1(x,\theta)\, C_{11}(\theta)\, z(\theta)\,dx\,d\theta \end{cases}$$

MINIMIZING THE EXPECTED COST (AS ON Pg. 1):

$$\frac{\pi_1 \int [C_{01}(\theta) - C_{11}(\theta)]\, z(\theta)\, p_1(x;\theta)\,d\theta}{\pi_0 [C_{11} - C_{00}]\, p_0(x)} \gtrless_{H_0}^{H_1} 1$$

- SPECIAL CASE: $C_{01}(\theta) = C_{01}$, $C_{11}(\theta) = C_{11}$

  DEFINE $\Lambda(x;\theta) = p_1(x,\theta)/p(x)$

  $\bar{\Lambda}(x) = \int \Lambda(x,\theta)\, z(\theta)\,d\theta$

  TEST IS: $\bar{\Lambda}(x) \gtrless_{H_0}^{H_1} \dfrac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$

  SPECIALLALAR CASE: TO MINIMIZE PROBABILITY OF ERROR, LET

  $C_{10} - C_{00} = 1$, $C_{01} - C_{11} = 1$ AND

  $\bar{\Lambda}(x) \gtrless_{H_0}^{H_1} \pi_0/\pi_1$

- A KARHUNEN-LOEVE INTEGRAL SOLUTION

FOR $R(t,s) = \min(t,s) \Leftarrow$ BROWNIAN MOTION

$$\lambda \phi(t) = \int_0^T R(t,s)\,\phi(s)\,ds$$
$$= \int_0^T \min(t,s)\,\phi(s)\,ds \quad (*)$$

$0 < s < t \Rightarrow \min(t,s) = s$

$t < s < T \Rightarrow \min(t,s) = t$

$$\Rightarrow \lambda \phi(t) = \int_0^t s\,\phi(s)\,ds + \int_t^T t\,\phi(s)$$

RECALL THAT:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\delta}{\delta t} F(x,t)\,dx + \frac{\delta b(t)}{\delta t} F[b(t),t] - \frac{\delta a(t)}{\delta t} F[a(t),t]$$

THUS: $\lambda \phi'(t) = t\phi(t) + \int_t^T \phi(s)\,ds - t\phi(t)$

$$= \int_t^T \phi(s)\,ds \quad (**)$$

AGAIN: $\lambda \ddot{\phi} = -\phi \Rightarrow \ddot{\phi}(t) + \frac{1}{\lambda}\phi(t) = 0$

{WE HAVE XFORMED INTEGRAL EQUATION
INTO A DIFFERENTIAL EQUATION

SOLN: $\phi(t) = A \sin\frac{t}{\sqrt{\lambda}} + B\cos\frac{t}{\sqrt{\lambda}}$

- FROM "$*$", $\phi(0) = 0 \Rightarrow B = 0$
- FROM "$**$", $\frac{d}{dt}\phi(T) = 0 \Rightarrow \cos\frac{T}{\sqrt{\lambda}} = 0$

$$\Rightarrow \lambda_n = \frac{T^2}{(n-\frac{1}{2})^2 \pi^2}$$

- REQUIRE $\int_0^T |\phi_n(t)|^2\,dt = 1$ (ORTHONORMALIZE)

GIVES: $A = \sqrt{\frac{2}{T}}$

RECALL: $R(t,s) = \sum_n \lambda_n \phi_n(t)\phi_n(s)$

THUS: $\min(t,s) = \frac{2T}{\pi^2}\sum_n \frac{1}{(n-\frac{1}{2})^2} \sin\frac{t}{\sqrt{\lambda_n}} \sin\frac{s}{\sqrt{\lambda_n}}$

SERIES CONVERGES POINTWISE

- UNIFORMLY MOST POWERFUL (UMP) TEST

$H_0 : \theta \in \Theta_0 \quad ; \quad H_1 : \theta \in \Theta_1$

$\alpha = \sup_{\theta \in \Theta_0} E_\theta \{\phi(x)\} = \alpha$ (sup ≡ LEAST UPPER BOUND)

- DEFN: A TEST OF SIZE α, $\phi_0(x)$, IS UMP IF $\forall$ OTHER TEST $\phi(x)$

$$E[\phi_0(x)] \geq E[\phi(x)] \quad \forall \quad \theta \in \Theta_1$$

- A UMP TEST EXISTS IFF THE N-P TEST CAN BE COMPLETELY DEFINED $\forall\ \theta$ WITHOUT KNOWLEDGE OF θ.

① EXAMPLE $H_0 : x = n \quad H_1 : x = n + \theta \quad ; \theta > 0$

$n \sim N(0, \sigma^2)$

THEN: $p_0(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$; $p_1(x;\theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$

$$\Lambda(x;\theta) = \exp\left[\frac{1}{2\sigma^2}(2x\theta - \theta^2)\right]$$

$$2x\theta - \theta^2 \underset{H_0}{\overset{H_1}{\gtrless}} T$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \tilde{T}$$

② EXAMPLE: SAME AS ABOVE, BUT $\theta < 0$

$$x \underset{H_1}{\overset{H_0}{\gtrless}} \tilde{T}$$

③ NO UMP TEST EXISTS FOR THIS CASE W/O KNOWLEDGE OF θ'S POLARITY

④ LAPLACE NOISE (SAME HYPOTHESIS AS ①)

$p_0(x) = = \frac{\gamma}{2} e^{-\gamma|x|}$, $p_1(x;\theta) = \frac{\gamma}{2} e^{-\gamma|x-\theta|}$

$$\Lambda(x) = e^{[-\gamma|x-\theta| + \gamma|x|]}$$

$$|x| - |x - \theta| \underset{H_0}{\overset{H_1}{\gtrless}} T$$

NO UMP TEST EXISTS

- MONOTONE LIKELIHOOD RATIO

IF $\Lambda = \frac{p(x,\theta_1)}{p(x,\theta_0)}$ IS NON-DECREASING IN X WHEN $\theta_0 < \theta_1$, THE REAL PARAMETER FAMILY DISTRIBUTION, $p(x;\theta)$, IS SAID TO HAVE A MONOTONE LIKELIHOOD RATIO.

- A UMP TEST: IF $p(x;\theta)$ HAS A MONOTONE LIKELIHOOD RATIO, THEN THE TEST

$$\phi(x) = \begin{cases} 1 & ; x > x_0 \\ p & ; x = x_0 \\ 0 & ; x < x_0 \end{cases}$$

IS UMP FOR TESTING $H_0 : \theta \leq \theta_0$ AGAINST $H_1 : \theta > \theta_0 \ \forall \ \theta_0 \in \Theta$

PROOF: DIRECTLY FROM N-P LEMMA;

$$\phi(x) = \begin{cases} 1 & p_1(x;\theta) > k\, p_0(x;\theta) \\ 0 & p_1(x;\theta) < k\, p_0(x;\theta) \end{cases}$$

THE TEST MAXIMIZES THE POWER, β, $\forall \ \theta$

- ONE PARAMETER EXPONENTIAL FAMILY (OPEF)

$p(x;\theta)$ COMES FROM A OPEF IFF IT CAN BE WRITTEN

$$p(x;\theta) = C(\theta)\, h(x) \exp[Q(\theta)\, T(x)]$$

THE CORRESPONDING LIKELIHOOD RATIO IS

$$\frac{p(x;\theta_1)}{p(x;\theta_0)} = \frac{C(\theta_1)}{C(\theta_0)} \exp[\{Q(\theta_1) - Q(\theta_0)\}\, T(x)]$$

- IF $Q(\theta)$ AND $T(x)$ ARE NON-DECREASING, THEN $p(x;\theta)$ HAS MONOTONE LIKELIHOOD RATIO
- EXAMPLE: GAUSSIAN: $p(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{+\theta x/2\sigma^2}$$

① THIS IS A OPEF:

$$C(\theta) = e^{-\theta^2/2\sigma^2}; \quad h(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

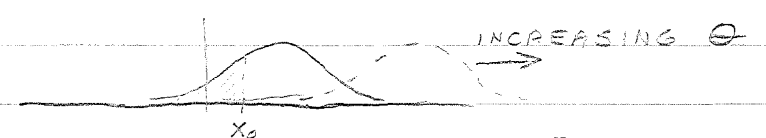
$$Q(\theta) = \frac{\theta}{2\sigma^2}; \quad T(x) = x$$

② $p(x;\theta)$ ALSO HAS A MONOTONE LIKELIHOOD RATIO

③ CONSIDER: $H_0: x = n + \theta \; ; \; \theta \le 0$

$H_1: x = n + \theta \; ; \; \theta > 0$

$$\alpha = \sup_{\theta \in \Theta_0} E_\theta\{\phi(x)\} = \sup_{\theta \le 0} \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}} dx$$



TO GET SUP (MAX), WE WANT θ AS SMALL AS POSSIBLE. $\theta = 0 \Rightarrow \alpha = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$

$$\Rightarrow x_0 = \sigma\, \Phi^{-1}(1-\alpha)$$

TEST OF SIZE α IS THUS

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \sigma\, \Phi^{-1}(1-\alpha) = x_0$$

NOTE THAT THE TEST IS UMP.

- SUFFICIENT STATISTIC
  - DEFN: LET X BE A RANDOM VARIABLE WHOSE DISTRIBUTION DEPENDS ON $\theta \in \Theta$. $T(x)$ IS A SUFFICIENT STATISTIC FOR θ, IF THE CONDITIONAL DISTRIBUTION OF X GIVEN THAT $T = t$, IS INDEPENDENT OF θ
  - FACTORIZATION THEOREM: $T(x)$ IS A SUFFICIENT STATISTIC IFF $\exists$ NON-NEGATIVE g AND h $\ni$ $p(x;\theta) = g(t(x),\theta)h(x)$
  - IN THE OPEF, $T(x)$ IS A SUFFICIENT STATISTIC. WE MAY WRITE $h(x) = h(x)$ AND $g[t(x),\theta] = C(\theta)\exp[Q(\theta)T(x)]$
  - EXAMPLE: $H_0: \theta \le \theta_0$ $\quad H_1: \theta > \theta_0$

    GIVEN 1. $p(x;\theta) = C(\theta)\,h(x)\exp[Q(\theta)T(x)]$

    2. $Q(\theta)$ IS NON-DECREASING

    3. HAVE K iid. SAMPLES

    $$p[x_1, x_2, \ldots x_K;\theta] = p(x_1;\theta)\,p(x_2;\theta)\ldots p(x_K;\theta)$$
    $$= [C(\theta)]^K[h(x_1)\ldots h(x_K)]\exp\Big[Q(\theta)\sum_{k=1}^{K}T(x_k)\Big]$$

    HERE, $Y = \sum_{k=1}^{K} T(X_k)$ IS SUFFICIENT STATISTIC

    TEST IS $\quad Y \underset{H_0}{\overset{H_1}{\gtrless}} T$

• EXAMPLE: FLIPPING A WEIGHTED COIN WITH UNKNOWN p

$H_0: \theta \le p \quad ; \quad H_1: \theta > p$

OBSERVING K iid FLIPS

$$P[N=n;\theta] = \binom{K}{n}\theta^n(1-\theta)^{K-n} \Leftarrow \text{BINOMIAL DISTRIBUTION}$$

$$= \binom{K}{n}(1-\theta)^K\left(\frac{\theta}{1-\theta}\right)^n$$

$$= \binom{K}{n}(1-\theta)^K e^{n\ln\left(\frac{\theta}{1-\theta}\right)}$$

$\ln\frac{\theta}{1-\theta}$ IS NON-DECREASING

$\Rightarrow P[N=n;\theta]$ HAS MONOTONE LIKELIHOOD RATIO

$\therefore n$ IS A SUFFICIENT STATISTIC

(= SUCCESSES IN K TRIALS)

TEST IS: $\phi(x) = \begin{cases} 1 & ; n > n_0 \\ p & ; n = n_0 \\ 0 & ; n < n_0 \end{cases}$

$$\alpha = \sup_{\theta \le p} E_\theta[\phi(x)]$$

$$= \sup_{\theta \le p} p\binom{K}{n_0}\theta^{n_0}(1-\theta)^{K-\theta_0} + \sum_{n_0+1}^{K}\binom{K}{n_0}\theta^n(1-\theta)^{K-n}$$

• EXAMPLE: LIFETIME TESTING (TIME OF OCCURANCE)

$P(x;\theta) = \theta e^{-\theta x} \quad ; x \ge 0 \quad , \theta > 0$

$H_0: \theta < t \quad ; \quad H_1: \theta \ge t \quad ; K$ iid SAMPLES

$$P(x_1, x_2, \ldots, x_K;\theta) = \theta^K e^{-\theta\sum_{n=1}^{K} x_n}$$

SUFFICIENT STATISTIC: $\sum_{n=1}^{K} x_n$

TEST IS $\sum_{n=1}^{K} x_n \gtrless_{H_1}^{H_0} T$ (NOTE INEQUALITIES)

CHOOSE $T \ni \alpha = \sup_{\theta \le 0} P_\theta\left[\sum_{n=1}^{K} x_n < T\right]$

• EXAMPLE: $H_0: \theta \le \theta_0 \quad H_1: \theta > \theta_0$

$$P(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta} \quad ; \theta > 0$$

K iid. SAMPLES

TEST IS $\sum_{K=1}^{K} x_K^2 \gtrless_{H_0}^{H_1} X_0^2$

- NON-OPTIMUM TESTING: IF UMP TEST DOESN'T EXIST, BEST WE CAN DO IS ESTIMATE Θ BY $\hat{\Theta}$ (FROM DATA) AND MAKE A N-P TYPE TEST:

$$\Lambda(x,\hat{\Theta}) \underset{H_0}{\overset{H_1}{\gtrless}} T \quad ; \quad \Lambda(x,\hat{\Theta}) = \frac{P_1(x,\hat{\Theta})}{P_0(x)}$$

MORE GENERALLY, IF $H_0: \Theta \in \Theta_0$, $H_1: \Theta \in \Theta_1$ THEN COMPUTE $\hat{\Theta}_0$ AND $\hat{\Theta}_1$ BE ESTIMATES ASSUMING H_0 AND H_1 RESPECTIVELY. THEN USE TEST $\frac{p(x,\hat{\Theta}_1)}{p(x,\hat{\Theta}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} T$

- UNBIASED ESTIMATE

$E[\hat{\Theta}(x)] = \int \hat{\Theta}(x)\, p(x;\Theta)\, dx = \Theta \Leftarrow$ UNBIASED

$E[\hat{\Theta}(x)] = \Theta + B \Leftarrow$ KNOWN BIAS (CHOOSE $\hat{\Theta}_N = \hat{\Theta} - B$)

$E[\hat{\Theta}(x)] = \Theta + B(\Theta) \Leftarrow$ UNKNOWN BIAS

- CRAMER-RAO INEQUALITY

$P_r[\,|\hat{\Theta}(x) - \Theta| < \lambda\,] \leq \frac{var[\hat{\Theta}(x)]}{\lambda^2} \Leftarrow$ CHEBYCHEV'S INEQUALITY

$E[\hat{\Theta}(x) - \Theta] = \int_{-\infty}^{\infty} [\hat{\Theta} - \Theta]\, p(x;\Theta)\, dx = 0 \Leftarrow$ FOR UNBIASED $\hat{\Theta}$

TAKE DERIVITIVE WRT Θ:

$$-\int_{-\infty}^{\infty} p(x;\Theta)\,dx + \int_{-\infty}^{\infty} [\hat{\Theta} - \Theta] \frac{d}{d\Theta} p(x;\Theta)\,dx = 0$$

SINCE $\int_{-\infty}^{\infty} p(x;\Theta)\,dx = 1$ AND $\frac{d}{d\Theta} p(x;\Theta) = p(x;\Theta)\frac{d}{d\Theta} \ln p(x;\Theta)$:

$$\int_{-\infty}^{\infty} [\hat{\Theta}(x) - \Theta]\, p(x;\Theta) \frac{\delta}{\delta\Theta} \ln p(x;\Theta)\, d\Theta$$

$$= \int \sqrt{p(x;\Theta)}\, \frac{\delta}{\delta\Theta} \ln p(x;\Theta) \times \sqrt{p(x;\Theta)}\, [\hat{\Theta}(x) - \Theta]\, dx = 1$$

USING SCHWARZ'S INEQUALITY:

$$\left|\int_{-\infty}^{\infty} f(t)\, g(t)\, dt\right|^2 \leq \int_{-\infty}^{\infty} |f(t)|^2 dt \int_{-\infty}^{\infty} |g(t)|^2 dt$$

GIVES $\int_{-\infty}^{\infty} p(x;\Theta) \left[\frac{\delta}{\delta\Theta} \ln p(x;\Theta)\right]^2 dx \int_{-\infty}^{\infty} p(x;\Theta)\, [\hat{\Theta}(x) - \Theta]^2 dx$

$$= E\left[\left(\frac{\delta}{\delta\Theta} \ln p(x;\Theta)\right)^2\right] var(\hat{\Theta}) \geq 1$$

$$\Rightarrow var[\hat{\Theta}(x)] \geq \frac{1}{E[\{\frac{d}{d\Theta} \ln p(x;\Theta)\}^2]} \Leftarrow \text{CRAMER-RAO INEQ}$$

$$E\left[\left\{\frac{\delta}{\delta\Theta} \ln p(x;\Theta)\right\}^2\right] = -E\left[\frac{\delta^2}{\delta\Theta^2} \ln p(x;\Theta)\right] \Leftarrow \text{ALTERNATE FORM}$$

IF $\hat{\Theta}$ IS BIASED ($E[\hat{\Theta}] = \Theta + B(\Theta)$), THEN

$$E[\{\hat{\Theta} - \Theta\}^2] \geq \left(1 + \frac{\delta B}{\delta\Theta}\right) \Big/ E\left[\left\{\frac{d}{d\Theta} p(x;\Theta)\right\}^2\right]$$

- PROOF OF $E\left[\left\{\frac{d}{d\theta} p(x;\theta)\right\}^2\right] = -E\left[\frac{d^2}{d\theta^2} p(x;\theta)\right]$

$$\int_{-\infty}^{\infty} p(x;\theta)\,dx = 1$$

$$\int_{-\infty}^{\infty} \frac{d}{d\theta} p(x;\theta)\,dx = \int_{-\infty}^{\infty} p(x;\theta) \frac{d}{d\theta} \ln p(x;\theta)\,dx = 0$$

$$\int_{-\infty}^{\infty} \frac{d^2}{d\theta^2} p(x;\theta)\,dx = 0 =$$

$$= \int_{-\infty}^{\infty} p(x;\theta) \left[\frac{d}{d\theta} \ln p(x;\theta)\right]^2 dx + \int_{-\infty}^{\infty} p(x;\theta) \frac{d^2}{d\theta^2} \ln p(x;\theta)$$

$$\Rightarrow \int_{-\infty}^{\infty} p(x;\theta) \left[\frac{d}{d\theta} \ln p(x;\theta)\right]^2 dx = -\int_{-\infty}^{\infty} p(x;\theta) \frac{d^2}{d\theta^2} \ln p(x;\theta)\,dx$$

THUS: $E\left[\left\{\frac{d}{d\theta} \ln p(x;\theta)\right\}^2\right] = -E\left[\frac{d^2}{d\theta^2} \ln p(x;\theta)\right]$

- EXAMPLE: $p(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\theta)^2/2\sigma^2}$

$$\left[\frac{d}{d\theta} \ln p(x;\theta)\right]^2 = \frac{(x-\theta)^2}{\sigma^4}$$

$$E\left\{\left[\frac{d}{d\theta} \ln p(x;\theta)\right]^2\right\} = E\left[\frac{(x-\theta)^2}{\sigma^4}\right] = \frac{1}{\sigma^2}$$

$\therefore$ FROM CRAMER-RAO: $E[(\hat{\theta}-\theta)^2] \geq \sigma^2$

WHERE $\hat{\theta}$ IS ANY UNBIASED ESTIMATOR OF θ

TRY $\hat{\theta} = \bar{X} \Rightarrow E[(\bar{X}-\theta)^2] = \sigma^2$

WE HAVE ACHEIVED C-R LOWER BOUND.

- EXAMPLE: $p(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}$; $\theta > 0$

$$\left[\frac{\delta}{\delta\theta} p(x;\theta)\right]^2 = \frac{x^4 - 2\theta x^2 + \theta^2}{4\theta^4}$$

SINCE $x \sim N(0,\sigma^2)$, $E[x^4] = 3\theta^4$

$$\Rightarrow E\left[\left\{\frac{\delta}{\delta\theta} p(x;\theta)\right\}^2\right] = \frac{1}{2\theta^2}$$

$\therefore$ FROM R-C: $E[\{\hat{\theta}-\theta\}^2] \geq 2\theta^2$

WHERE $\hat{\theta}$ IS AN UNBIASED ESTIMATE

- EFFICIENT ESTIMATE: AN UNBIASED ESTIMATE WHOSE VARIANCE ACHEIVES THE R-C. LOWER BOUND.
  - NECESSARY AND SUFFICIENT CONDITION:

    $\hat{\theta}$ IS EFFICIENT IFF $\frac{\delta}{\delta\theta} \ln p(x;\theta) = K(\theta)[\hat{\theta} - \theta]$

    PROOF: ON pg. 13, SCHWARZ'S INEQUALITY IS EQUAL IFF

    $$\sqrt{p(x;\theta)}\ \frac{\delta}{\delta\theta} \ln p(x;\theta) = K(\theta)\sqrt{p(x,\theta)}\ [\hat{\theta}(x) - \theta]$$

    WHERE $K(\theta)$ IS A PROPORTIONALITY CONSTANT.

    $\hat{\theta}(x)$ MUST SATISFY THIS CONDITION TO BE EFFICIENT.

    THUS: $\frac{\delta}{\delta\theta} \ln p(x;\theta) = K(\theta)\ [\hat{\theta}(x) - \theta]$ Q.E.D.
  - EXAMPLE: $p(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\theta)^2/2\sigma^2}$

    $\frac{\delta}{\delta x} p(x;\theta) = \frac{x-\theta}{\sigma^2}$. NOTE: X IS UNBIASED

    $\Rightarrow \hat{\theta}(x) = X$ IS AN EFFICIENT ESTIMATE OF θ
  - EXAMPLE: $p(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}$; $\theta > 0$

    $\frac{\delta}{\delta\theta} p(x;\theta) = \frac{x^2-\theta}{2\theta^2}$

    $E[x^2] = 0 \Rightarrow x^2$ IS UNBIASED

    $\therefore X^2$ IS AN EFFICIENT ESTIMATE OF θ

- MAXIMUM LIKELIHOOD ESTIMATION

CHOOSE Θ THAT MAXIMIZES $p(x;\Theta)$

EQUIVALENTLY, SOLVE $\frac{d}{d\Theta} p(x;\Theta) = \frac{d}{d\Theta} \ln p(x;\Theta) = 0$

$0 = \frac{\delta}{\delta\Theta} \ln p(x;\Theta) \Rightarrow$ LIKELIHOOD EQUATION.

CALL ROOTS OF THE EQUATION $\hat{\Theta}_{me}$

- RELATION TO EFFICIENT ESTIMATORS

$$\frac{\delta}{\delta\Theta} \ln p(x;\hat{\Theta}_{me}) = 0 = K(\Theta)\left[\hat{\Theta}_{me} - \hat{\Theta}\right]$$

$\therefore$ IF AN EFFICIENT ESTIMATOR EXISTS, IT IS $\hat{\Theta}_{me}$

- PROPERTIES OF $\hat{\Theta}_{me}$

1. IF $\hat{\Theta}_{me}$ IS NOT EFFICIENT, WE REALLY DON'T KNOW HOW GOOD IT IS
2. $\hat{\Theta}_{me}$ IS MAXIMUM LIKELIHOOD ESTIMATE OF Θ.
   $f(\Theta)$ IS A MONOTONE FUNCTION OF Θ.
   $\Rightarrow f(\hat{\Theta}_{me})$ IS A MAXIMUM LIKELIHOOD ESTIMATE OF Θ.
3. FOR $\mathbb{K}$ INDEPENDENT SAMPLES, $\hat{\Theta}_{me} = \hat{\Theta}_{me}(x_1, x_2, \ldots, x_{\mathbb{K}})$.
   UNDER SOME LOOSE CONDITIONS, THE FOLLOWING HOLD
   a. CONSISTANCY: $\hat{\Theta}_{me}$ IS A CONSISTANT ESTIMATE OF Θ:
   $$\forall \epsilon > 0, \lim_{\mathbb{K}\to\infty} P\left[\,|\hat{\Theta}_{me} - \Theta_o| > \epsilon\right] = 0$$
   b. ASYMPTOTICALLY UNBIASED
   $$\lim_{\mathbb{K}\to\infty} E\left[\hat{\Theta}_{me}\right] = \Theta_o$$
   c. ASYMPTOTICALLY EFFICIENT
   $$\lim_{\mathbb{K}\to\infty} E\left[\{\hat{\Theta}_{me} - \Theta_o\}^2\right] = \frac{1}{E\left[\{\frac{\delta}{\delta\Theta}\ln p(x;\Theta)\}^2\right]} = \mathrm{Var}\,\Theta_{me}$$
   d. ASYMPTOTICALLY GAUSSIAN: AS $\mathbb{K}\to\infty$, $\hat{\Theta}_{me} \sim N(\Theta_o, \mathrm{Var}\,\hat{\Theta}_{me})$

- RELATIONSHIP TO SUFFICIENT STATISTIC

IF $T(x)$ IS A SUFFICIENT STATISTIC FOR θ,

THEN $\hat{\theta}_{ml}$ MUST BE A FUNCTION OF T

$$p(x,\theta) = g\left[T(x),\theta\right]h(x)$$
$$\frac{\delta}{\delta\theta}\ln p(x;\theta) = \frac{\delta}{\delta\theta}\ln g\left[T(x),\theta\right]$$

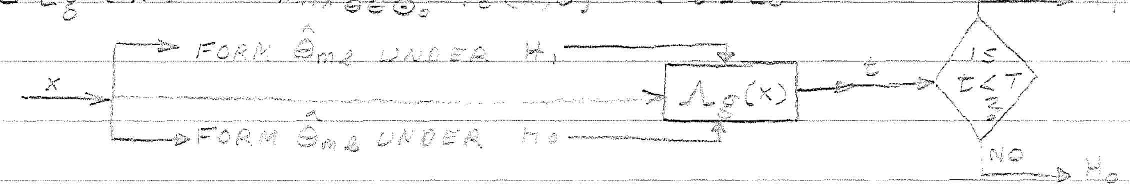
- MAXIMUM LIKELIHOOD DETECTOR

$H_0: p_0(x;\theta)\ ;\ \theta\in\Theta_0$

$H_1: p_1(x;\theta)\ ;\ \theta\in\Theta_1$

ASSUME UMP DOESN'T EXIST. WILL USE A MAXIMUM LIKELIHOOD PROCEDURE (GENERALIZED LIKELIHOOD RATIO TEST)

$$\Lambda_g(x) = \frac{\max_{\theta\in\Theta_1} p_1(x,\theta)}{\max_{\theta\in\Theta_0} p_0(x,\theta)} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0$$



EXAMPLE: $H_0: x = n\ ;\ H_1: x = \theta + n\ ,\quad n\sim N(0,\sigma^2)$

K iid SAMPLES $\Rightarrow \hat{\theta}_{ml} = \bar{x} = \frac{1}{K}\sum_{k=1}^{K} x_k$

$$\ln\Lambda_g(x_1,x_2,\dots,x_K) = \frac{1}{\sigma^2}\sum_{k=1}^{K}\left(x_k\bar{x} - \frac{1}{2}\bar{x}^2\right)$$

GIVES $\left(\sum_{k=1}^{K} x_k\right)^2 = 2K\sigma^2\ln\Lambda_g$

TEST IS $\left(\sum_{k=1}^{K} x_k\right)^2 \underset{H_0}{\overset{H_1}{\gtrless}} T^2$

$$\alpha = P_0\left[\left(\sum_{k=1}^{K} x_k\right)^2 > T^2\right] \Leftarrow \text{NOTE: NOT OPTIMAL}$$
$$= 2\,\Phi\left(\frac{-T}{\sqrt{K}\sigma}\right) \Rightarrow T = -\sqrt{K}\,\sigma\,\Phi^{-1}\left(\frac{\alpha}{2}\right) > 0$$

$$\beta_{ml}(\theta) = P_1\left[\left(\sum_{k=1}^{K} x_k\right)^2 > T^2\right] = \beta(-\theta)$$
$$= 1 + \Phi\left[\frac{-T-K\theta}{\sqrt{K}\sigma}\right] - \Phi\left[\frac{T-K\theta}{\sqrt{K}\sigma}\right]$$
$$= 1 + \Phi\left[\Phi^{-1}\left(\frac{\alpha}{2}\right) - \frac{\theta}{\sigma}\sqrt{K}\right] - \Phi\left[-\Phi^{-1}\left(\frac{\alpha}{2}\right) - \frac{\theta}{\sigma}\sqrt{K}\right]$$

IF WE KNEW θ'S POLARITY, UMP POWER WOULD BE

$$\beta(\theta) = 1 - \Phi\left[\Phi^{-1}(1-\alpha) - \frac{|\theta|}{\sigma}\sqrt{K}\right]$$

BELEIVE IT OR NOT, FOR $\frac{\sqrt{K}|\theta|}{\sigma}$ LARGE, $\beta(\theta) \approx B_{ml}(\theta)$

NOTE: $B_{ml}(\theta) \le \beta(\theta)$

- CONTINUOUS TIME DETECTION IN GAUSSIAN NOISE

$H_0: x(t) = n(t) \qquad H_1: x(t) = n(t) + \theta s(t)$

$s(t) = \int_a^b R(t,s)\, q(s)\, ds \qquad q_k = \frac{s_k}{\lambda_k}$

RECALL, THAT FOR NON-COMPOSITE HYPOTHESIS

$$\ln \Lambda = \sum_k \left(\frac{x_k s_k}{\lambda_k} - \frac{1}{2}\frac{s_k^2}{\lambda_k} \right) = \sum (x_k q_k - \tfrac{1}{2} s_k q_k)$$

$$= \int_a^b x(t) q(t)\, dt - \frac{1}{2}\int_a^b s(t) q(t)\, dt \quad (1)$$

FOR COMPOSITE CASE, A θ IS ASSOCIATED WITH q AND s:

$$\ln \Lambda(\theta) = \theta \int_a^b x(t) q(t)\, dt - \frac{\theta^2}{2}\int_a^b s(t) q(t)\, dt$$

⊗ NOTE: IF POLARITY OF θ IS KNOWN, WE GOTTA UMP TEST

ASSUME POLARITY OF θ IS NOT KNOWN.

Eq. 1 IS OF FORM: $f(\theta) = U\theta - \frac{1}{2}V\theta^2 \Rightarrow f(\frac{U}{V}) \geq f(\theta) \ \forall \theta$

$$\Rightarrow \hat{\theta}_{me} \left(= \frac{U}{V}\right) = \int_a^b x(t) q(t)\, dt \Big/ \int_a^b s(t) q(t)\, dt$$

- $\hat{\theta}_{me}$ UNBIASED:

$$E_1[\hat{\theta}_{me}] = \frac{E[\int_a^b (n(t) + \theta s(t))\, q(t)\, dt]}{\int_a^b s(t) q(t)\, dt} = \theta$$

- $\hat{\theta}_{me}$ IS EFFICIENT

$$\frac{\delta}{\delta\theta} \ln \Lambda(\theta) = \int_a^b x(t) q(t)\, dt - \theta \int_a^b s(t) q(t)\, dt$$

$$= \frac{\delta}{\delta\theta} \ln p_1(x;\theta)$$

$$= \int_a^b x(t) q(t)\, dt\, [\hat{\theta}_{me} - \theta]$$

- $\therefore$ var$(\hat{\theta}_{me})$ ACHEIVES C-R BOUND

UNDER H_1: $x(t) = \theta s(t) + n(t)$

$$\Rightarrow E_1\left[\left(\frac{\delta}{\delta\theta} \ln \Lambda(x(t),\theta)\right)^2\right] = E_1\left[\int_a^b n(t), q(t)\, dt\right]^2$$

$$= \int_a^b s(t) q(t)\, dt$$

$$= d^2 = SNR = \{\text{var}[\hat{\theta}_{me}]\}^{-1}$$

MAY SHOW THIS ANOTHER WAY

$$\text{var}[\hat{\theta}_{me}] = E_1[(\theta - \hat{\theta}_{me})^2]$$

$$= E_1\left[\left\{\theta - \frac{\int_a^b (\theta s(t) + n(t))\, q(t)\, dt}{\int_a^b s(t) q(t)\, dt}\right)^2\right]$$

$$= 1/d^2$$

- THE DETECTOR

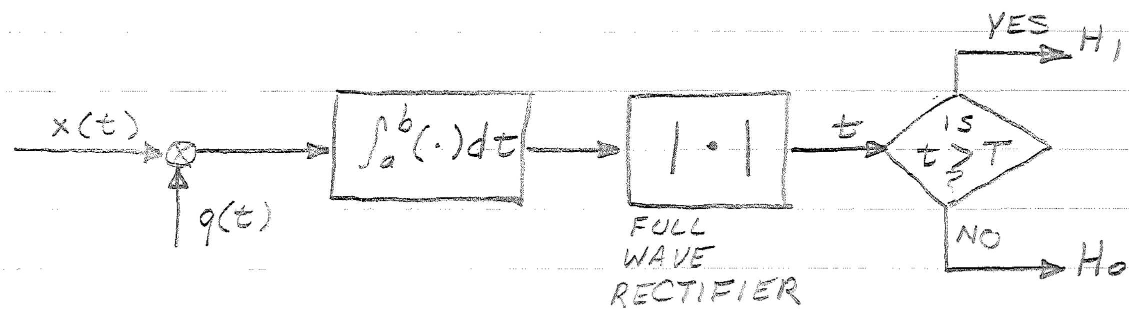
$$\ln \Lambda[x(t), \hat{\theta}_{ml}] = \frac{\left[\int_a^b x(t)\, q(t)\right]^2}{2\int_a^b q(t)\, s(t)\, dt}$$

RECALL $\int_a^b q(t)\, s(t)\, dt = \Sigma q_k s_k = \Sigma \frac{s_k}{\lambda_k} > 0$

THUS DETECTOR IS

$$\left[\int_a^b x(t)\, q(t)\, dt\right]^2 \gtrless_{H_0}^{H_1} T^2$$

OR $\left|\int_a^b x(t)\, q(t)\, dt\right| \gtrless_{H_0}^{H_1} T > 0$



LOOSE ENDS PLUG SHEET: FINAL

- NON-PARAMETRIC DETECTION

$H_0: \; p = \frac{1}{2}$

$H_1: \; p > \frac{1}{2}$

K iid. SAMPLES

$$\begin{cases} f(\cdot) = \text{DENSITY OF ANY } X_i \text{ UNDER } H_1 \\ f^+(\cdot) = \text{ " " " " " " GIVEN } X_i > 0 \\ f^-(\cdot) = \text{ " " " " " " GIVEN } X_i < 0 \end{cases}$$

$$\Rightarrow f(x) = p f^+(x) + (1-p) f^-(x) \quad \text{(UNDER } H_1)$$

AND $f_0(x) = \frac{1}{2}\left[f^+(x) + f^-(x) \right]$ (UNDER H_0)

CONSIDER N-P. TEST: $H_0: f_0$, $H_1: f$

$$\Lambda = \prod_{k=1}^{K} f(x_k) / f_0(x_k)$$

$$f(x_k)/f_0(x_k) = \begin{cases} p f^+ / \frac{1}{2} f^+ = 2p & ; x_k > 0 \\ (1-p) f^- / \frac{1}{2} f^- = 2(1-p) & ; x_k < 0 \end{cases}$$

THEN $\Lambda(\vec{x}) = 2^K \cdot p^{\sum \mu(x_k)} (1-p)^{K - \sum \mu(x_k)}$

$$= 2^K (1-p)^K \left(\frac{p}{1-p} \right)^{\sum \mu(x_k)}$$

SO TEST IS: $\sum_{k=1}^{K} \mu(x_k) \underset{f_0(\cdot)}{\overset{f(\cdot)}{\gtrless}} T \Leftarrow$ SIGN DETECTOR

THIS CONCLUSION HOLDS FOR ALL f & f_0. THUS

$\sum_{k=1}^{K} \mu(x_k) \underset{H_0}{\overset{H_1}{\gtrless}} T$ IS N-P. OPTIMAL

- TEST STATISTIC

$$D = \sum_{k=1}^{K} \mu(x_k)$$

D IS DISTRIBUTED BINOMIALLY

UNDER H_0: $P[D = d] = \binom{K}{d} \left(\frac{1}{2}\right)^d \left(1 - \frac{1}{2}\right)^{K-d} = \left(\frac{1}{2}\right)^K \binom{K}{d}$

UNDER H_1: $P[D = d] = \binom{K}{d} p^d (1-p)^{K-d}$

$$\alpha = \sum_{d=T+1}^{K} \binom{K}{d} \left(\frac{1}{2}\right)^K \quad \text{(MAY ALSO RANDOMIZE)}$$

$$\beta = \sum_{d=T+1}^{K} \binom{K}{d} p^d (1-p)^{K-d}$$

ASYMPTOTIC RELATIVE EFFICIENCY (ARE)

- DEFN: DETECTORS D_1 D_2

$N_i(\alpha, \beta)$ = # SAMPLES D_i NEEDS TO ACHEIVE α & β

$ARE_{2,1}$ = ARE OF D_2 W.R.T. D_1

$$= \lim_{\substack{N_1 \to \infty \\ N_2 \to \infty \\ H_1 \to H_0}} \frac{N_1(\alpha, \beta)}{N_2(\alpha, \beta)}$$

- EX: D_2 = SIGN DETECTOR D_1 = LINEAR (GAUSSIAN) DETECTOR

FOR D_2 (FROM CENTRAL LIMIT THEOREM)

UNDER H_0: $D_2 \sim \binom{K}{d}\left(\frac{1}{2}\right)^K \sim N\left(\frac{K}{2}, \frac{K}{4}\right)$

UNDER H_1: $D_2 \sim \binom{K}{d} p^d (1-p)^{K-d} \sim N(Kp, Kp(1-p))$

THEN: $$\alpha_2 = \int_T^\infty \frac{1}{\sqrt{2\pi}\sqrt{K/4}} e^{-(x-K/2)^2/(2\times K/4)} dx$$
$$= \Phi\left[\frac{K-2T}{\sqrt{K}}\right]$$
$$\beta_2 = \int_T^\infty \frac{1}{\sqrt{2\pi}\sqrt{Kp(1-p)}} e^{-\frac{(x-Kp)^2}{2Kp(1-p)}} dx$$
$$= \Phi\left[\frac{Kp}{\sqrt{Kp(1-p)}}\right]$$
$$= \Phi\left[\frac{\sqrt{K}(2p-1) - \Phi^{-1}(1-\alpha_2)}{2\sqrt{p(1-p)}}\right]$$

FOR D_1: UNDER H_0: $E_0(X_K) = 0$

UNDER H_1: $E_0(X_K) = \mu > 0$ (ASSUMING MEDIUM=MEAN)

THEN $$\alpha_1 = \int_T^\infty \frac{1}{\sqrt{2\pi K}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$
$$= 1 - \Phi\left[\frac{T}{\sqrt{K}\sigma}\right]$$
$$\beta_1 = \int_T^\infty \frac{1}{\sqrt{2\pi K}\sigma} e^{-\frac{(x-K\mu)^2}{2\sigma^2}} dx$$
$$= \Phi\left[\frac{K\mu - T}{\sqrt{K}\sigma}\right]$$
$$= \Phi\left[\frac{\mu}{\sigma}\sqrt{K_1} - \Phi^{-1}(1-\alpha_1)\right]$$

$$\beta_1 = \beta_2 \Rightarrow \frac{\mu}{\sigma}\sqrt{K_1/K_2} = \frac{\Phi^{-1}(1-\alpha_1)}{\sqrt{K_2}} + \frac{(2p-1)}{2\sqrt{p(1-p)}} - \frac{\Phi^{-1}(1-\alpha_2)}{2\sqrt{K_2 p(1-p)}}$$

$$\frac{K_1}{K_2} \xrightarrow[K_2 \to \infty]{K_1 \to \infty} (p-\tfrac{1}{2})^2\sigma^2 / \mu^2 p(1-p)$$

$$\therefore ARE_{2,1} = \lim_{H_1 \to H_0} \frac{(p-\frac{1}{2})^2\sigma^2}{\mu^2 p(1-p)}$$

FOR SYMMETRIC DENSITIES: $p = P[X_K \leq 0] = \int_{-\infty}^0 f(x)dx = \frac{1}{2} + \int_0^\mu f(x-\mu)dx$

AS $H_1 \to H_0$, $\mu \to 0$ AND $p \to \frac{1}{2} + \mu f(0)$

$$\Rightarrow \therefore ARE_{2,1} = \lim_{\mu \to 0} \frac{f^2(0)\sigma^2}{\frac{1}{4} - \mu^2 f^2(0)} = 4\sigma^2 f^2(0)$$

- EX: TAKE NOISE TO BE GAUSSIAN:

$$f(0) = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$\Rightarrow ARE_{2,1} = \frac{2}{\pi} \cong 0.64$$

- EX: TAKE NOISE TO BE LAPLACIAN

$$f(x) = \frac{\gamma}{2} e^{-\gamma|x|} \Rightarrow \sigma^2 = 2/\gamma^2$$

$$f(0) = \frac{\gamma}{2}$$

$$\Rightarrow ARE_{2,1} = 2$$

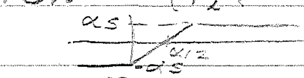
$\therefore$ SIGN DETECTOR IS TWICE AS GOOD AS LINEAR

NOTE: $ARE_{2,1} > 1$ IF $4\sigma^2 f^2(0) > 1$

- BAYES TEST: $P_e = \pi_0 Q_0 + \pi_1 Q_1$: $\Lambda(Y) \gtrless_{H_0}^{H_1} \frac{\pi_0}{\pi_1}$
- WEIGHTED COST CRITERION: $J = \pi_0 Q_0 C_0 + \pi_1 Q_1 C_1$: $\Lambda(Y) \gtrless \frac{\pi_0 C_0}{\pi_1 C_1}$
- GENERAL BAYES CRITERION: $R = \pi_0 Q_0 (C_{10} - C_{00}) + \pi_1 Q_1 (C_{01} - C_{11})$

$$\Lambda(Y) \gtrless \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

- MINIMAX CRITERION: $R(\pi_1) = \pi_0 [C_{00} Q_0 + C_{10}(1 - Q_0)] + \pi_1 [C_{11} Q_1 + C_{01}(1 - Q_1)]$

  GIVES $C_{01}(1 - \hat{Q}_1) + C_{11}\hat{Q}_1 = C_{00}\hat{Q}_0 + C_{10}(1 - \hat{Q}_0) \Rightarrow RIST = C_{00} Q_0 + C_{10}(1 - Q_0)$
- LOCALLY OPTIMAL (WHITE NOISE): $g_{LO} = -\frac{d}{dx} \ln f(x)$
- WHITE NOISE DETECTION $\Rightarrow \{Y_i\} \rightarrow \boxed{ZML} \rightarrow \boxed{\Sigma} \rightarrow \boxed{?}$
- LAPLACE NOISE ZML  ; $p = \frac{\alpha}{2} e^{-\alpha |x|}$
- MERCER'S THEM: $R(t,s) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t) \phi_n(s)$; $\lambda_n \phi_n(t) = \int R(t,s) \phi_n(s)$
- PSEUDO SIGNAL: $s(t) = \int R(t,s) q(s) ds$
- $SNR = d^2 = \int s(t) q(t) = \sum s_k q_k = \sum \frac{s_k^2}{\lambda_k} = \frac{(s,q)^2}{E[(n,q)^2]}$
- TEST STATISTIC: $G = \int Y q$ $\quad E_0(G) = 0,\ E_1(G) = d^2,\ \mathrm{var}\, G = d_k^2$
- BAYES COMPOSITE H: $\frac{\pi_1 \int [C_{01}(\theta) - C_{11}(\theta)]\, p(x,\theta)\, \Xi(\theta)\, d\theta}{\pi_0 (C_{10} - C_{00})\, p(\theta)} \gtrless 1$
- UMP TEST: T w/o θ ; $\alpha = \sup_{\theta \in \Theta_0} E_0[\phi(x)]$
- MONOTONE L.R.: $\Lambda = \frac{p(x;\theta_1)}{p(x;\theta_0)}$ NON-DECR. $\forall\ \theta_1 > \theta_0$
- ONE PARAMETER EXPONENTIAL FAMILY: $p(x;\theta) = h(x) C(\theta) e^{Q(\theta) T(x)}$
- SUFFICIENT STATISTIC: $p(x,\theta) = g[T(x), \theta]\, h(x)$
- UNBIASED: $E(\hat{\theta}) = \theta$
- CRAMER RAO INEQ: $\mathrm{var}(\hat{\theta}) \geq \frac{1}{E[\{\frac{\delta}{\delta\theta} \ln p(x;\theta)\}^2]}$
- EFFICIENT: $\frac{\delta}{\delta\theta} \ln p(x;\theta) = K(\theta)[\hat{\theta} - \theta]$
- MAXIMUM LIKELIHOOD: $\frac{\delta}{\delta\theta} \ln p(x;\theta) \big|_{\theta = \theta_{ml}} = 0$
- NON-PARAMETRIC TEST STATISTIC: $D = \sum \mu(x_i)$
- $ARE_{2,1} = \lim_{\substack{N_2 N_1 \to \infty \\ H_1 \to H_0}} \frac{N_1(\alpha_1, \beta)}{N_2(\alpha_2, \beta_2)}$ $\quad = 4\sigma^2 f^2(0)$: GAUS., SIGN DET